Advanced Macro - Lecture 1

The Solow Model and the Data

Ref: Acemoglu ch. 1-3, Barro and Sala-i-Martin ch. 1

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Aim and Scope of the Course

- two main goals:

  1. provide a rigorous theoretical framework to think about economic growth and development

  2. introduction to modern (dynamic) macroeconomics

     develop the workhorse models and tools that are at the core of macroeconomics

- strong emphasis on methods

Plan of the Course

1. first look at the data + Solow Model

2. towards the neoclassical growth model

   - key microfoundations

   - dynamic optimization in continuous and discrete time

3. infinite-horizon neoclassical growth model

4. overlapping generations

5. human capital and growth

6. technical progress and endogenous growth

Pre-requisites

- this is an advanced course

- mathematical tools required:

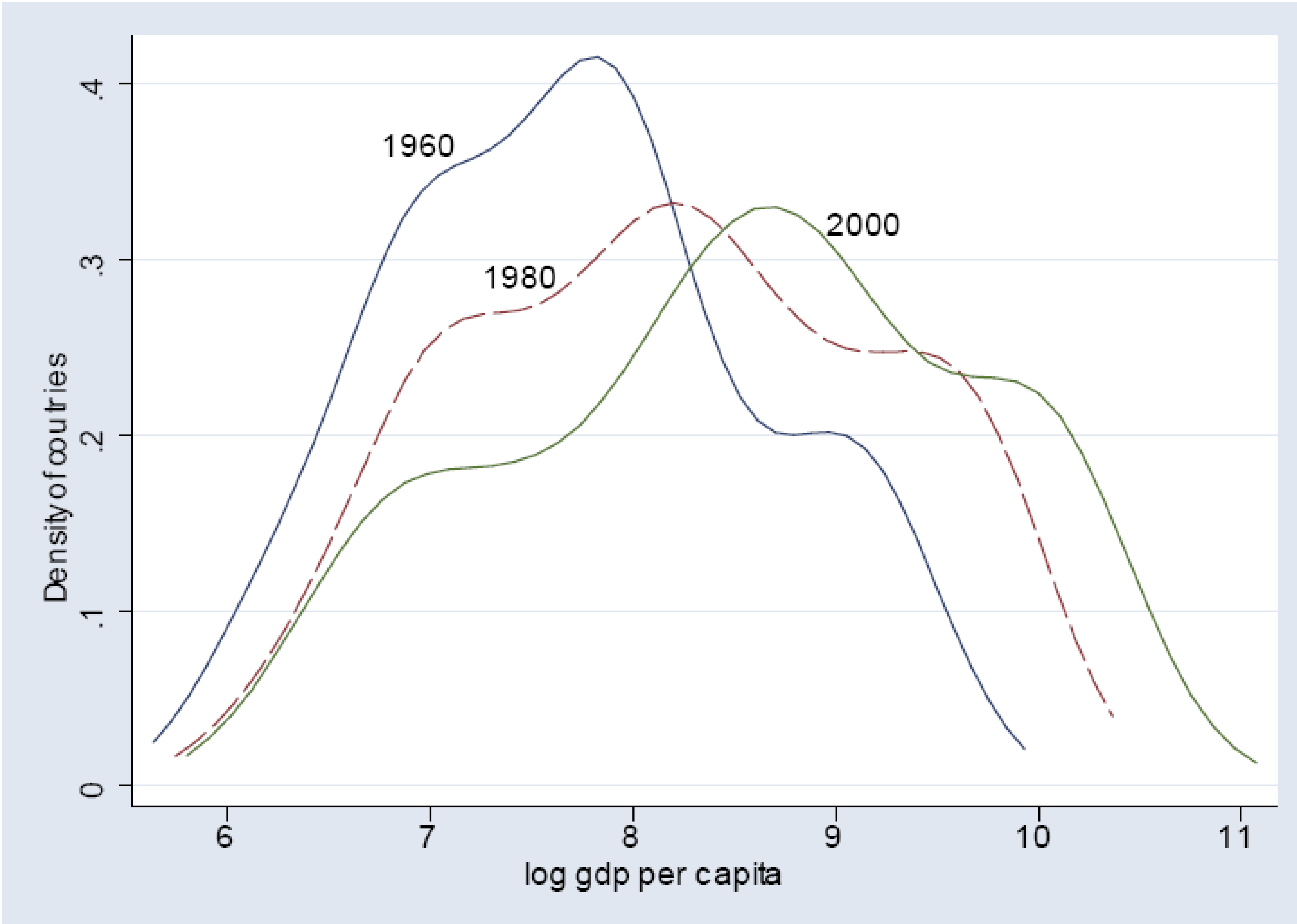
  1. calculus

  2. static optimization

  3. basic knowledge of difference/differential equations

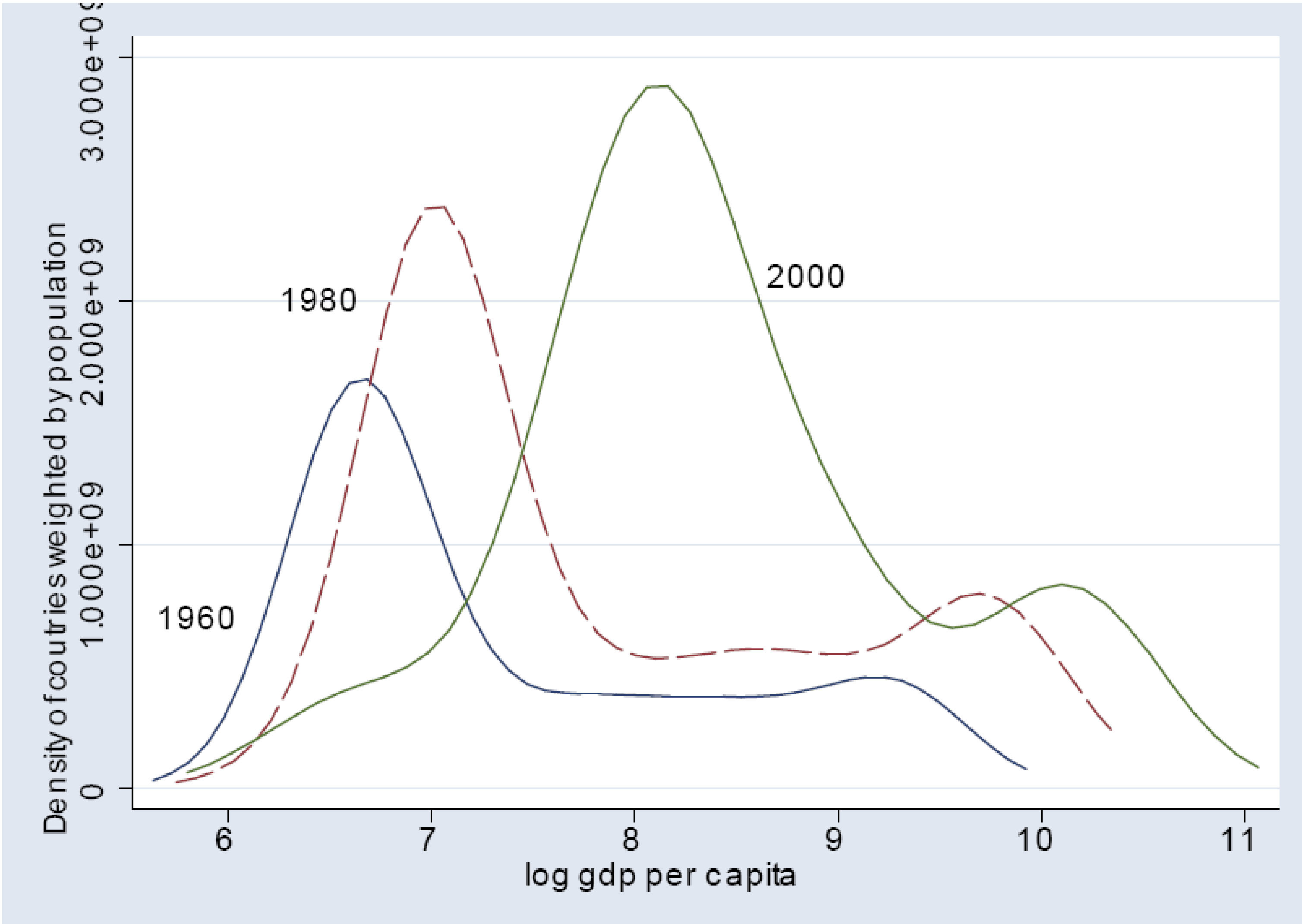
- dynamic maximization will be studied along the way

The World Income Distribution I



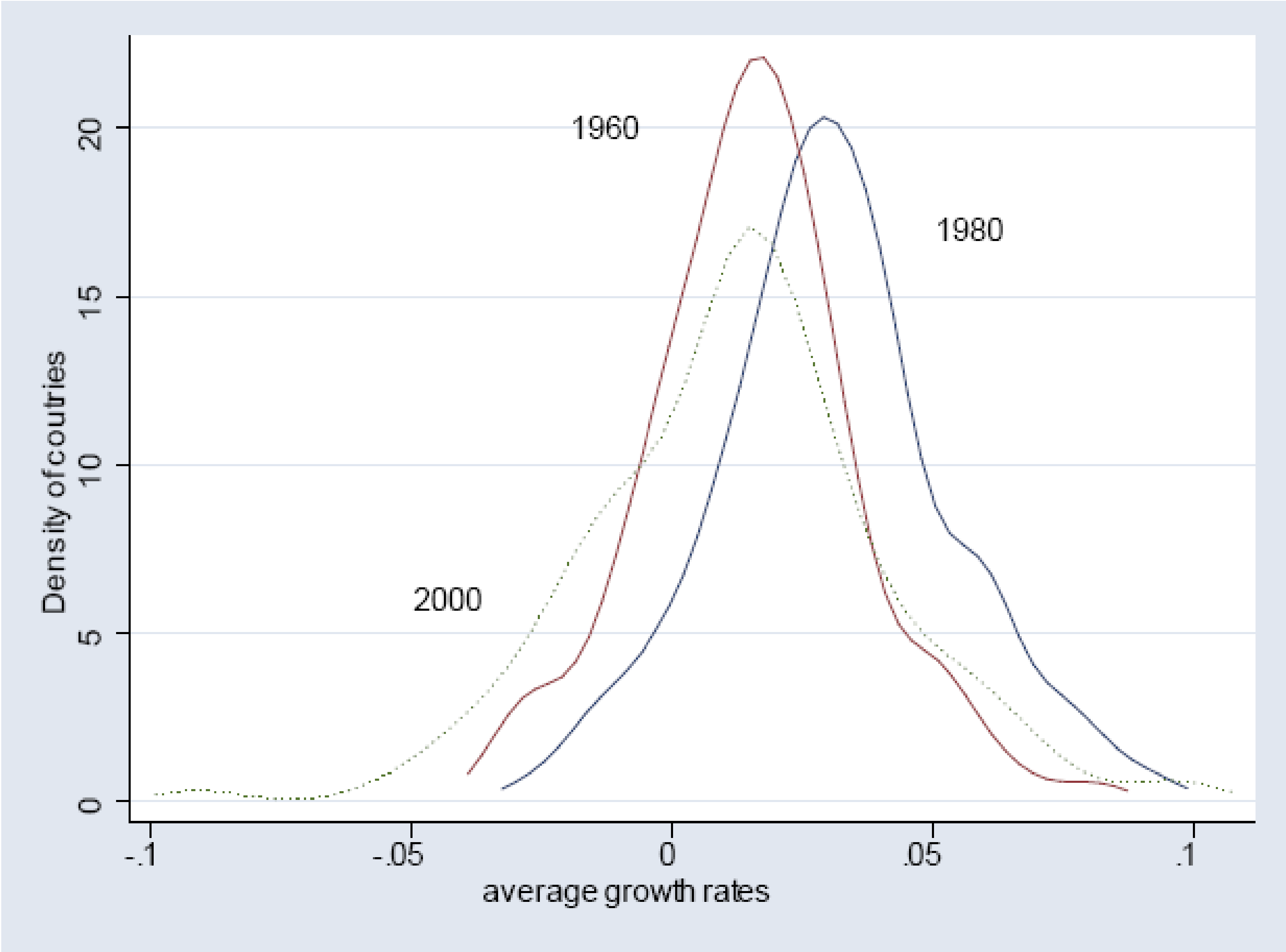
some divergence...

The World Income Distribution II

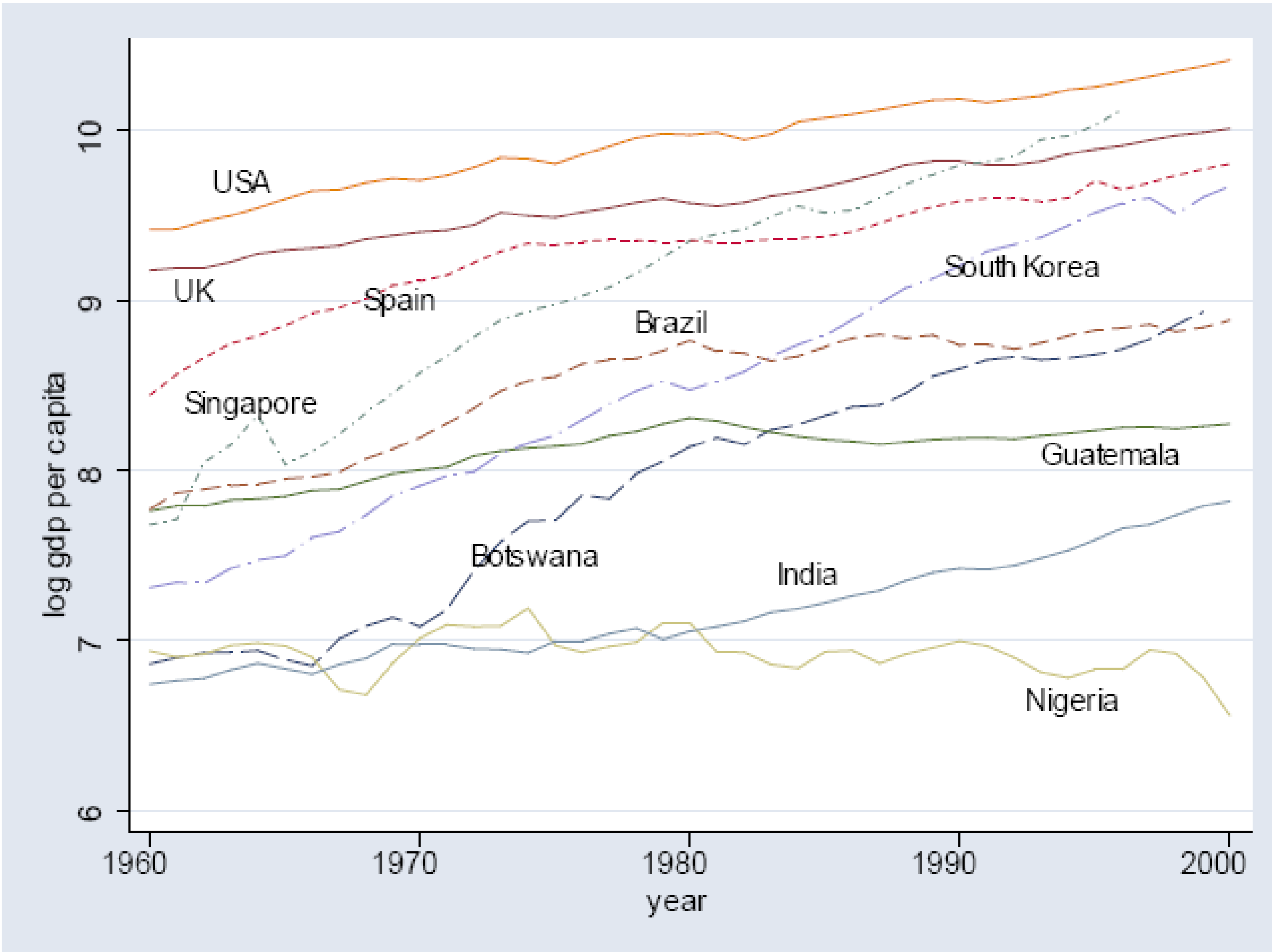


but not for the population-weighted distribution (why? India and China)

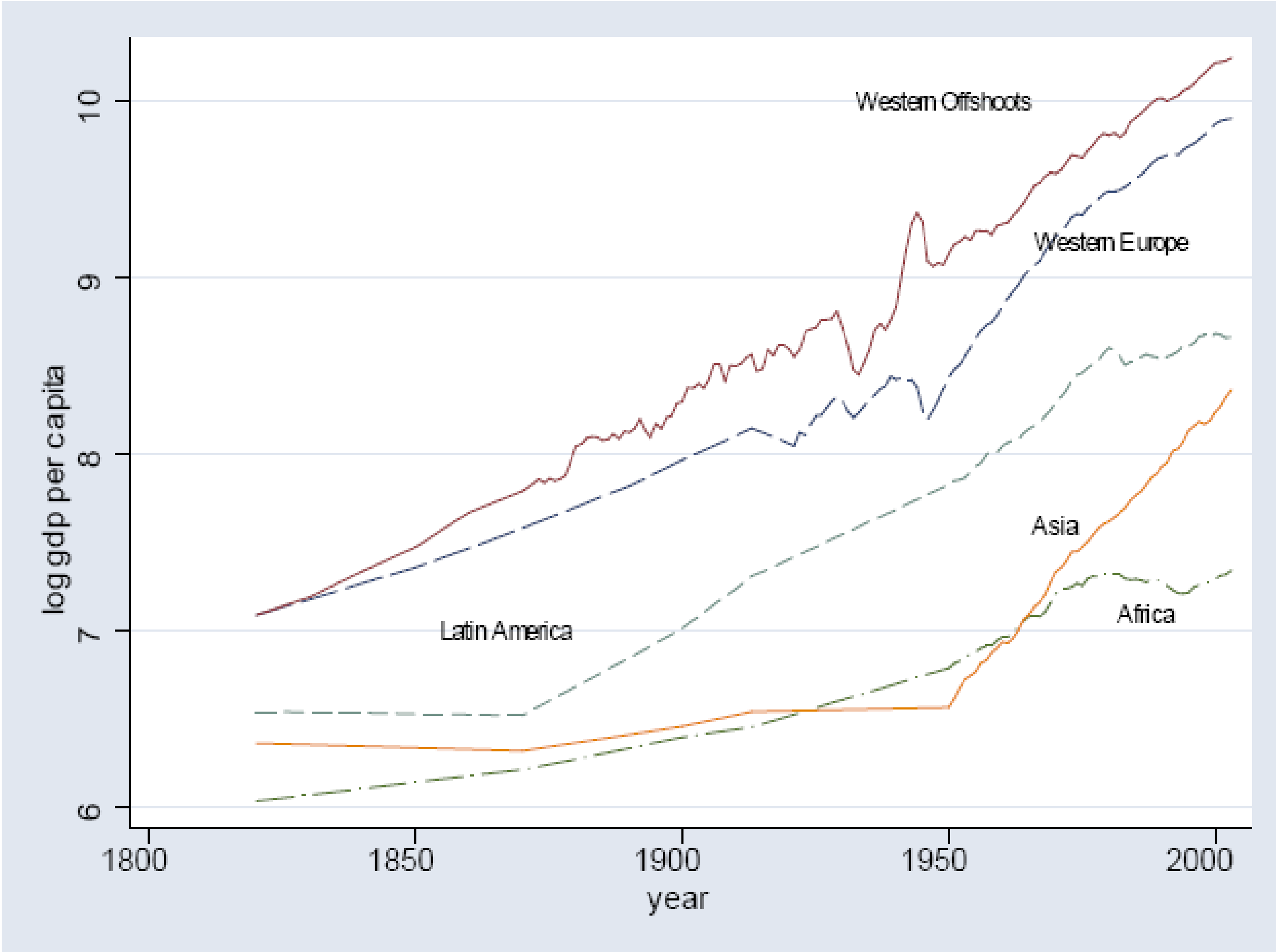
Growth Patterns



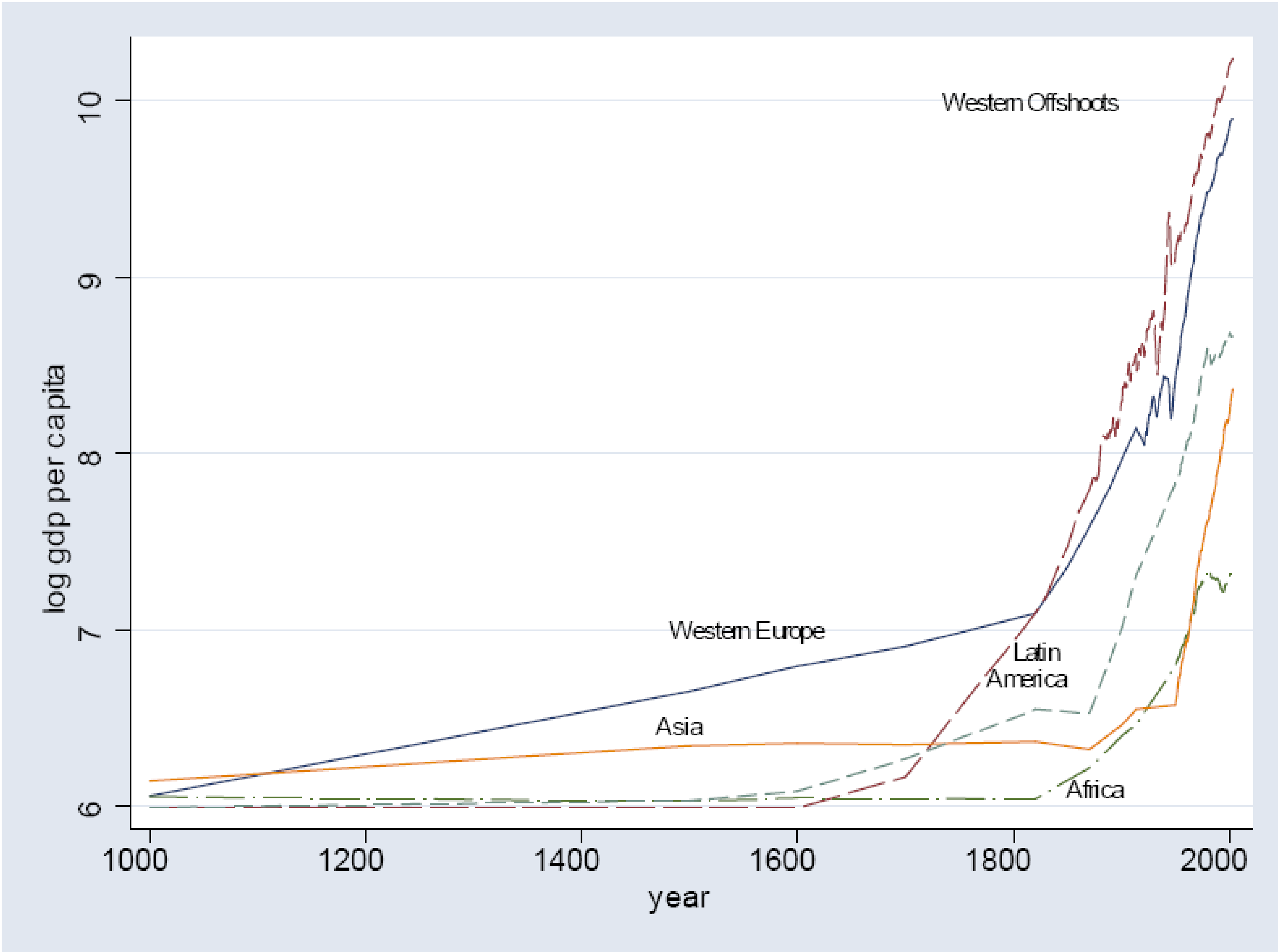
Growth in Selected Countries



Origin of Growth and Income Differences I



Origin of Growth and Income Differences II



Economic Growth: Some Key Questions

- why are some countries so much richer than others?
- how can the US economy grow at a steady rate?
- how did countries like Singapore and South Korea manage to catch-up?
- why do some counties (e.g. Nigeria) do not grow at all?

The Solow Model

- simple dynamic model useful for:

  1. thinking about the mechanics of economic growth

  2. accounting for cross-country income differences

- closed economy, unique final good (numeraire)

- exogenous saving rate s

- population grows at the exogenous rate n : $\frac{\dot{L}_t}{L_t} = n$

- core of the Solow model: neoclassical aggregate production function

The Neoclassical Production Function (no technical progress)

$$Y_t = F(K_t, L_t, t) = F(K_t, L_t)$$

- increasing and concave

$$F_K > 0 \quad F_{KK} < 0 \quad F_L > 0 \quad F_{LL} < 0$$

- homogeneous of degree one in K and L (Constant Returns to Scale):

$$F(\lambda K_t, \lambda L_t) = \lambda F(K_t, L_t) \quad \text{for all } \lambda > 0$$

- Inada conditions:

$$\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty$$

$$\lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0$$

- intensive form: $y_t \equiv \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) \equiv f(k_t)$

The Fundamental Equation

- change in K_t is equal to investment minus depreciation at the rate δ :

$$\dot{K}_t = I_t - \delta K_t = sY_t - \delta K_t$$

  where we have used $I_t = sY_t$

- in per capita units: $k_t = \frac{K_t}{L_t}$
  take logs, time differentiate and use $\dot{K}_t$:

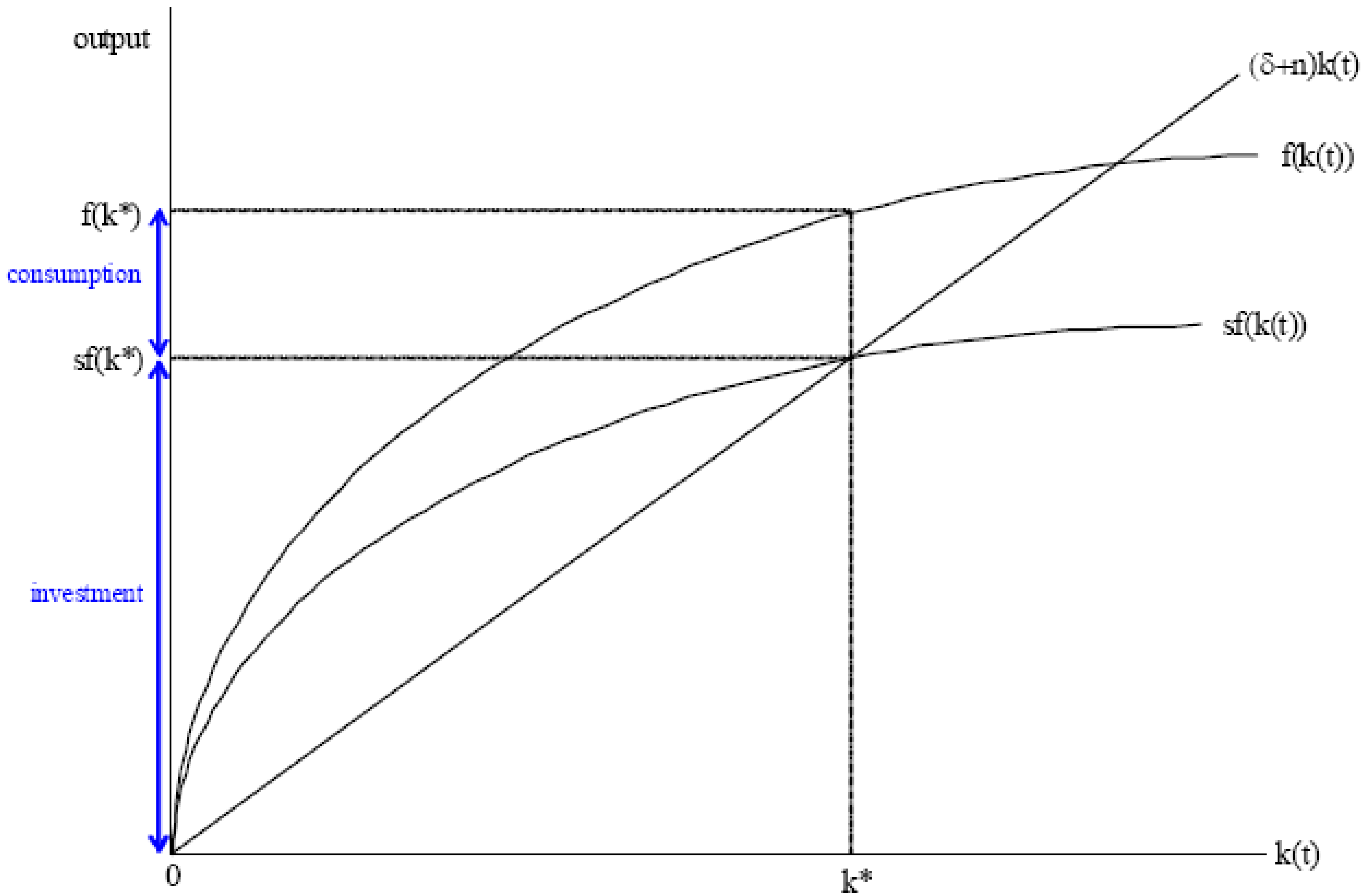
$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = s\frac{Y_t}{K_t} - \delta - n$$

  rearranging and using $\frac{Y_t}{L_t} = f(k_t)$

$$\dot{k}_t = sf(k_t) - (\delta + n) k_t$$

- a non-linear differential equation that can be studied graphically

The Solow Model: Graph



Steady-State

- a steady-state equilibrium is an equilibrium path in which $k_t = k$ for all t

- the Solow model tends to the steady state equilibrium over time

- set $\dot{k}_t = 0$:

$$sf(k) = (\delta + n) k$$

- the steady-state investment is equal to the amount of capital needed to make up for depreciation and to provide newborn with k

Steady-State Consumption and the Golden Rule

- the fraction $(1 - s)$ of disposable income is consumed:

$$c_t = (1 - s) f(k_t) = f(k_t) - sf(k_t)$$

- in steady-state, $sf(k) = (\delta + n) k$:

$$c = f(k) - (\delta + n) k$$

  note: c is an inverted U shaped function of k

- what is the level k^g that maximizes c?

$$f'(k^g) = \delta + n$$

- above k^g consumption can be increased by saving less (dynamic inefficiency)

Transitional Dynamics and Convergence

- along the transition, the growth rate of k :

$$\frac{\dot{k}_t}{k_t} = s\frac{f\left(k_t\right)}{k_t} - \delta - n$$

substitute s from the steady-state relationship, $s = \frac{(\delta+n)k}{f(k)}$:

$$\begin{aligned}
\frac{\dot{k}_t}{k_t} &= s\frac{f\left(k_t\right)}{k_t} - \delta - n = \frac{\left(\delta + n\right)k}{f\left(k\right)}\frac{f\left(k_t\right)}{k_t} - \delta - n \\
\frac{\dot{k}_t}{k_t} &= \left(\delta + n\right)\left[\frac{f\left(k_t\right)/k_t}{f\left(k\right)/k} - 1\right] = \left(\delta + n\right)\left[\frac{f\left(k_t\right)/k_t}{\kappa} - 1\right]
\end{aligned}$$

- Note: the average product of capital, $f\left(k_t\right)/k_t$, falls as we approach the steady-state
$\longrightarrow$ countries far from their steady state accumulate more and grow faster

- Conditional Convergence:
among countries converging to the same steady-state, poor countries grow faster

The Solow Model with Exogenous Technical Progress

- suppose now that labor productivity grows exogenously over time:

$$Y_t = F\left(K_t, A_t L_t\right)$$

with

$$\dot{A}_t / A_t = x$$

- $A_t L_t =$ effective labor (labor augmenting technical progress)

- now, define variables per effective units of labor:

$$\widehat{k}_t = \frac{K_t}{A_t L_t} \quad \text{and} \quad \widehat{y} = \frac{Y\left(K_t, A_t L_t\right)}{A_t L_t} = f(\widehat{k}_t)$$

take logs and time-differentiate $\widehat{k}_t$:

$$g_{\widehat{k}} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} - \frac{\dot{A}_t}{A_t} = \frac{\dot{K}_t}{K_t} - n - x$$

where $g_x = \dot{x}/x$.

The Fundamental Equation with Technical Progress

- substitute $\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} - \delta$:

$$g_{\widehat{k}} = s\frac{Y_t}{K_t} - \delta - n - x = sf(\widehat{k}_t)/\widehat{k}_t - (\delta + n + x)$$

- converges to a steady-state with $g_{\widehat{k}} = 0$:

$$sf(\widehat{k})/\widehat{k} = \delta + n + x$$

- since $\widehat{k} = k_t/A_t$ and $\widehat{y} = y_t/A_t$ are constant (in steady-state):

$$\frac{\dot{k}^{ss}}{k^{ss}} = \frac{\dot{y}^{ss}}{y^{ss}} = \frac{\dot{c}^{ss}}{c^{ss}} = x$$

  all per capita variables grow at the rate of technical progress

The Solow Model and the Data

- the Solow model motivates the following empirical approaches:

  1. test for convergence

  2. test of the steady-state predictions with regression analysis

  3. use the neoclassical production function to perform accounting exercises

Unconditional Convergence: No

Conditional Convergence (OECD countries): Yes

Testing the Steady-State equation

- assume $Y_t = K_t^{\beta} (A_t L_t)^{1-\beta}$

- straightforward to show that the steady-state is:

$$\widehat{k} = \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\beta}} \rightarrow Y_t = \left(\frac{s}{n+\delta+g}\right)^{\frac{\beta}{1-\beta}} A_t L_t$$

- Mankiw, Romer and Weil (1992) estimate the Y_t/L_t eq. using cross-country data on s_j and n_j:

$$\ln y_j = const + \frac{\beta}{1-\beta} \ln\left(s_j\right) - \frac{\beta}{1-\beta} \ln\left(n_j + \delta + g\right) + \varepsilon_j$$

- key assumptions:
  1. $\delta + g = 0.05$ same for all countries
  2. A_j uncorrelated to other variables and included in the error term

MRW (1992): Results

Estimates of the Basic Solow Model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	1.42	1.01	1.22
	(.14)	(.11)	(.13)
$\ln(n+g+\delta)$	-1.97	-1.12	-1.31
	(.56)	(.55)	(.36)
Adj R^2	.59	.49	.49
Implied β	.59	.50	.55
No. of observations	98	98	107

MRW (1992): Further Results and Comments

- implied $\beta \simeq 2/3$ while available data suggest the capital share to be around $1/3$

- MRW (1992) also augment the model with human capital (like physical capital, with exogenous investment rate s^h)
  Results:
  implied $\beta \simeq 1/3$, higher R^2
  $\longrightarrow$ technology differences not so important

- Problems:
  1. A_j is correlated with investment rates: s^k and s^h may pick up the effect of technology differences
  2. the implied human capital share is too high compared to available micro evidence

Productivity Accounting

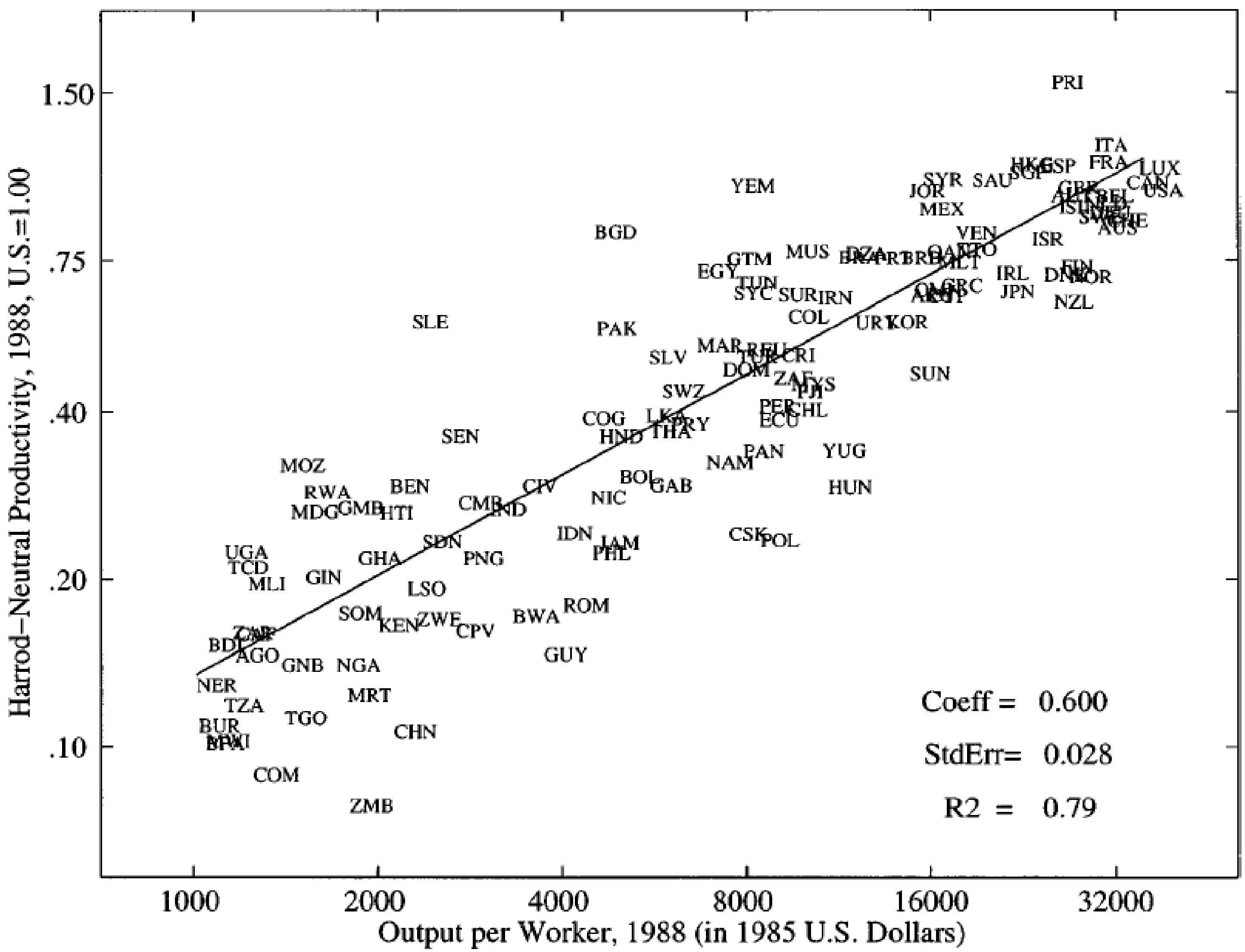
- postulate an aggregate production function

$$Y_j = K_j^{1/3} \left(A_j H_j\right)^{2/3}$$

- using cross-country (j) data on output, physical and human capital we can measure technology, A_j, as a residual

- typical results (Hall and Jones 1999, Caselli, 2005):
  1. A_j varies a lot across countries
  2. variation in technology $\left(A_j\right)$ explain around $1/2$ of the variation in GDP

Hall and Jones (1999)



Summary

- understanding economic growth is one of the most difficult/important tasks in macroeconomics

- the Solow model is a useful description of the growth process
  - it is mechanical
  - it does not allow for welfare analysis
  - yet, it suggests a number of interesting ways to look at the data

- main empirical results:
  - conditional convergence
  - technology differences are very important in explaining income differences

Advanced Macro - Lecture 2

Microfoundations of Growth Theory

Ref: Acemoglu (2009) ch. 5

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Towards the Neoclassical Growth Model

- limit of the Solow model: exogenous saving rate

- cannot be used to study the determinants of the saving rate

- cannot be used for normative purposes (no notion of welfare)

- by introducing complete preference ordering we overcome both limitations:

  1. saving decisions as outcome of individual utility-maximizing choices

  2. individual preferences allow to study the welfare property of the equilibrium (Pareto efficiency) and if there is scope for welfare improving intervention

Agents

- unit measure $h \in [0, 1]$ of households
  $\longrightarrow$ each households is infinitesimal
  $\longrightarrow$ aggregates and averages are the same

- time horizon of households:
  1. infinitely-lived (or overlapping generations linked through altruism)
  2. finite planning horizon (overlapping generations with no altruism)

- households have preferences orderings over *all* commodities (utility function)

- e.g.: over consumption levels (c_t) over all dates

- to make progress, we impose a number of restrictions on preferences

Time Separable Preferences

- preferences have two components:

  1. an instantaneous utility function (felicity), e.g. $u_h(c_t)$ with $u_h : \mathbb{R}_+ \rightarrow \mathbb{R}$ increasing and concave
     Note 1: $u_h(\cdot)$ is *independent* of time
     Note 2: no consumption externalities across households (envy?) and time (habits?)

  2. time discounting: instantaneous utility at a later date is less valuable
     typically, we assume exponential discounting

- assuming discrete time + infinite horizon, household preferences at $t = 0$ are:

$$\sum_{t=0}^{\infty} \beta_h^t u_h(c_t), \quad \beta_h \in (0, 1) \tag{1}$$

  where β_h is the discount factor of household h

- households will make choices in order to maximize (1)

Time Consistency

- time separability $+$ exponential discounting ensure "time-consistency"

- an action plan $\{x_t\}_{t=0}^{T}$ (T may be ∞) is time-consistent when:

  - if $\{x_t^*\}_{t=0}^{T}$ is optimal starting at $t = 0$

  - then $\{x_t^*\}_{t=t'}^{T}$ must be the optimal plan starting at t'

- no reason to revise the optimal plan at later dates

The Representative Household

- so far, u_h and β_h are household-h specific

- despite this heterogeneity, an economy admits a representative household if the preference side can be represented as if a single household made the aggregate decisions subject to a single budget constraint

- a "normative" representative household exists if we can also use the utility function of the representative household for welfare comparisons

- trivial case: when households are identical

- under some conditions, a representative household exists despite underlying heterogeneity in preferences/income

The Representative Household: Failures

- in general, a representative household does NOT exists

- **Debreu-Mantel-Sonnenschein Theorem**
  assume there are $N < \infty$ (with $N \in \mathbb{N}_+$) commodities and H households
  denote p a vector of prices
  assume that $x(p) : p \to \mathbb{R}^N_+$ is any continuous function that satisfies Walras' Law and is homogeneous of degree 0
  then, there exists an exchange economy where the aggregate demand is given by $x(p)$

- thus, aggregate demand can be almost arbitrary! (why? income effects)

- yet, under some restriction often taken for granted, the representative household does exist

The Representative Household: Conditions

- **Gorman's Aggregation Theorem**
  assume there are $N < \infty$ (with $N \in \mathbb{N}_+$) commodities and $H < \infty$ households suppose that preferences of household h can be represented by an indirect utility function of the form

$$v_h(p, y_h) = a_h(p) + b(p)\, y_h$$

  (NOTE: $b(p)$ is common for all h) then there exists a representative household with indirect utility

$$v_h(p, y_h) = \int_{h \in H} a_h(p) + b(p) \int_{h \in H} y_h$$

  moreover, the representative household can be used for normative purposes

- all is required is that there exists a monotonic transformation of the indirect utility function that takes the Gorman form

- many commonly-used preferences satisfy this condition

Aggregation: Intuition

- aggregate demand can be written as: $x = \int_{h \in H} x_h(p, y_h)$

- aggregation holds when x is the demand function of a single consumer with income $\int_{h \in H} y_h$:

$$x = x\left(p, \int_{h \in H} y_h\right)$$

- for this to be the case we must have, for all goods and for all income redistributions with $\int_{h \in H} dy_h = 0$:

$$\int_{h \in H} \frac{\partial x_h(p, y_h)}{\partial y_h} dy_h = 0$$

in turn requiring:

$$\frac{\partial x_h(p, y_h)}{\partial y_h} = \frac{\partial x_{h'}(p, y_{h'})}{\partial y_{h'}} \longrightarrow \text{ same income effects}$$

so that individual demand changes from any income redistribution cancel out

Example: CES Preferences

- suppose

$$u_h(c_t) = \left[\sum_{j=1}^{N} \left(c_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

  with $\sigma \in (0, \infty)$.

- different households may have different income.

- static maximization program:

$$\mathcal{L} = u_h(c_t) + \lambda \left(y_h - \sum_{j=1}^{N} p_j c_j\right)$$

  FOCs imply:

$$\frac{c_j}{c_i} = \left(\frac{p_i}{p_j}\right)^{\sigma}$$

Example: CES Preferences Continued

- to find $v_h(p, y_h)$, multiply by p_j, sum over all j :

$$\begin{aligned}
\frac{p_j c_j}{c_i} &= \left(p_j\right)^{1-\sigma}(p_i)^{\sigma} \\
\frac{y_h}{c_i} &= (p_i)^{\sigma}\sum_{j=1}^{N}\left(p_j\right)^{1-\sigma} \\
c_i &= \left(\frac{1}{p_i}\right)^{\sigma}\frac{y_h}{\sum_{j=1}^{N}\left(p_j\right)^{1-\sigma}}
\end{aligned}$$

- substitute c_i into $u_h(\cdot)$:

$$v_h(p, y_h) = \left[\sum_{j=1}^{N}\left(p_j\right)^{1-\sigma}\right]^{\frac{1}{\sigma-1}} y_h$$

- satisfies the Gorman form.

Infinite Planning Horizon

- many growth/macro models assume that individuals have an infinte-planning horizon

- two microfoundations

1. Poisson death probability $= \nu$:

$$\begin{aligned} U_{t=0} &= u(c_0) + \beta(1-\nu)u(c_1) + \beta^2(1-\nu)^2 u(c_2) + ... \\ &= \sum_{t=0}^{\infty} [\beta(1-\nu)]^t u(c_t) \end{aligned}$$

   utility in case of death $= 0$

   $\beta(1-\nu) =$ effective discount factor

2. finite life $+$ intergenerational altruism

Intergenerational Altruism

- individuals live for one period and have a single offspring
- "pure altruism:" individuals care about the utility of their offspring (with some discount factor β)
- then the utility of the individual is

$$\begin{aligned} U_t &= u(c_t) + \beta U_{t+1} \\ U_{t+1} &= u(c_{t+1}) + \beta U_{t+2} \\ &\rightarrow U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) \end{aligned}$$

- what if individuals live more than one period? Might run into time inconsistency!

Discrete vs. Continuous Time

- assume the economy admits a representative household

- objective function in discrete time:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $\beta \in (0, 1)$ is the discount factor

- in continuous time, this utility function becomes:

$$\int_0^{\infty} e^{-\rho t} u(c_t)\, dt$$

  where $\rho > 0$ is the discount rate

Exponential Discounting

- value of 1 util in T periods from now, when the discount rate is ρ for each sub-period Δt :

$$V_T = (1 - \Delta t \rho)^{T/\Delta t}$$

usual compound interest formula with a negative interest rate

- now let $\Delta t \to 0$:

$$V_T = \lim_{\Delta t \to 0} (1 - \Delta t \rho)^{T/\Delta t}$$

take logs

$$\ln V_T = \lim_{\Delta t \to 0} \frac{\ln (1 - \Delta t \rho)}{\Delta t / T} = \lim_{\Delta t \to 0} \frac{-T\rho}{1 - \Delta t \rho} = -T\rho$$

where I have used L'Hospital rule. Thus:

$$V_T = e^{-\rho T}$$

The Representative Firm

- production side of the economy: let F be the set of firms

- aggregate production set of the economy:

$$Y = \left\{ \sum_{f \in F} y_f \in Y_f \right\}$$

  where $Y_f \in \mathbb{R}^N$ is the convex production possibility set of firm f

- **Representative Firm Theorem:**
  1. no production externalities
  2. competitive markets

  $\rightarrow$ there exists a representative firm

- thus, the representative firm assumption is without loss in generality. Why? there are no income effects in production

Competitive Equilibria (CE)

- an **economy** is described by preferences u, endowments ω, production sets Y, consumption sets X and profit shares θ

- a **competitive equilibrium** (CE) is given by an allocation (x^*, y^*) and a price vector p^* such that

  1. the allocation is feasible

  2. for every firm $f \in F$, y^* maximizes profits

  $$p^* y^* \geq p^* y \ \text{ for all } y$$

  3. for every consumer $h \in H$, x^* maximizes utility

Welfare Theorems I

- connection between Pareto optima and competitive equilibria

- **First Welfare Theorem**:
  Suppose that (x^*, y^*, p^*) is a CE of the economy $(H, F, u, \omega, Y, X, \theta)$ with H finite. Assume that all households are non-satiated. Then, (x^*, y^*) is Pareto optimal.

- intuition:
  1. utility maximization $\longrightarrow$ if another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable
  2. profit maximization $\longrightarrow$ any competitive equilibrium already maximizes the set of affordable allocations

- note:
  1. powerful normative result: a CE (if it exists) cannot be improved upon
  2. based on very few assumptions (no convexity required!)

Welfare Theorems II

- the converse of the first welfare theorem

- **Second Welfare Theorem**:
  consider a Pareto optimal allocation (x^*, y^*). Suppose that all production and consumption sets are convex and all utility functions are continuous and quasi-concave and satisfy non-satiation. Then, there exists a price vector p^* and an endowment and profit share allocations (ω^*, θ^*) such that (x^*, y^*, p^*) is a competitive equilibrium.

- thus, any Pareto optimal allocation can be replicated by a CE

- corollary: since Pareto optimal allocations can be decentralized as CE, a competitive equilibrium must *exist*

- this is why the second theorem is much more demanding: it contains an existence argument

Optimal and Competitive Growth

- in sum:

$$\text{competitive equilibria} \Longleftrightarrow \text{Pareto Optima}$$

- useful application in growth theory:
  second welfare theorem + representative agent assumption $\longrightarrow$
  we can look for the optimal growth allocation that maximizes the utility of the representative household and assert that this will correspond to the competitive equilibrium

- useful shortcut because the optimal growth allocation (social planner solution) is often simpler to characterize

Sequential Trading I

- in Arrow-Debreu general equilibrium models, all households trade at $t = 0$ irrevocable claims to commodities indexed by date and state of nature.

- growth models often assume sequential trading: separate markets (for labor, goods, assets) at each t

- yet, with complete markets (and time consistent preferences), both are equivalent

Sequential Trading II

- if agents have access to a basic Arrow security (bonds) allowing them to transfer resources across different dates

- they can just transfer throughout time the required amount of income, and then use it to buy commodities at spot markets

- this is because household correctly anticipate all future prices

- the role of these bonds is often played by the capital stock

- if the price of one good (aggregate output) is normalized to one in each period, $\longrightarrow$ the interest rate becomes the only relevant intertemporal price

Optimal Growth in Discrete Time

- dynamic optimization program:

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} st \quad : \quad & k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \\ & k_0 \quad \text{given} \end{aligned}$$

  where $u(c_t)$ is the utility of the representative agent, $\beta \in (0,1)$ the discount factor, k_t is the capital stock, $f(k_t)$ is the aggregate production function and $\delta \in (0,1)$ is the depreciation rate.

- solution will be two difference equations $\longrightarrow$ two boundary conditions are needed

- k_0 provides one initial condition, the second one will be the transversality condition (part of the solution)

- equivalently, we can solve for the competitive equilibrium

CE Growth in Discrete Time

- representative household program:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} st \quad : \quad & a_{t+1} = (1 + r_t) a_t + w_t - c_t \\ & a_0 \quad \text{given} \end{aligned}$$

where a_t is asset holding, r_t is the interest rate and w_t the equilibrium wage - from profit maximization $r_t = f_K(k_t)$ and $w_t = f(k_t) - r_t k_t$

- next, we need market clearing: $a_t = k_t$

- note: in discrete time there are different conventions on the timing of interest payment (beginning/end of period)

The Flow Budget Constraint

- NOTE:
  the flow constraint $a_{t+1} = (1 + r_t)\, a_t + w_t - c_t$ is not sufficient
  why? we need to make sure debt does not explode (no Ponzi scheme)

- solutions:

  1. use the lifetime budget constraint:
     present discounted value of income = present discounted value of consumption

  2. add a limiting condition (such as a lower bound to a_t or the No-Ponzi game condition) and keep the flow constraint (simpler)

Optimal Growth in Continuous time

- dynamic optimization program:

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c_t)\, dt$$

$$\begin{aligned} st \quad : \quad & \dot{k}_t = f(k_t) - \delta k_t - c_t \\ & k_0 \quad \text{given} \end{aligned}$$

  where $\rho > 0$ is the discount factor

- next classes: review and apply the mathematical tools of dynamic optimization:
  1. dynamic programming in discrete time
  2. optimal control in continuous time

Summary

- dynamic optimization and GE theory are at the core of modern macroeconomics

- to make progress we need to impose structure
  1. time-separability
  2. exponential discounting
  3. representative household
  4. no consumption externalities

- under further assumptions (regularity conditions + competitive markets and no production externalities) the Welfare Theorems imply that the competitive equilibrium exists and coincides with the social planner solution

Advanced Macro - Lecture 3

Introduction to Dynamic Programming

Ref: Acemoglu (2009) ch. 6

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Discrete-Time Infinite Horizon Optimization

- typical problem:

$$\begin{aligned} & \max_{\{y_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, y_t) \\ st \quad : \quad & y_t \in \widetilde{G}(x_t) \\ : \quad & x_{t+1} \in \widetilde{f}(x_t, y_t), \quad x_0 \text{ given} \end{aligned}$$

  $\beta \in (0,1] =$ discount factor
  $x_t \in X \subset \mathbb{R}^{K_x}$: state variables (predetermined, e.g. capital)
  $y_t \in Y \subset \mathbb{R}^{K_y}$: control variables (choice variables)
  $U : X \times Y \to \mathbb{R}$ instantaneous payoff
  $\widetilde{G}(x_t)$ constraint on admissible controls, for given state
  $\widetilde{f}(x_t, y_t)$ law of motion of the state

- sometimes convenient to substitute y_t as a function of x_{t+1} and x_t

Stationary Sequence Problem

- Sequence Problem A1:

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$$
$$st \quad : \quad x_{t+1} \in G(x_t), \qquad x_0 \text{ given}$$

- $V(x_t)$ is a real-valued function

- what is the control now? x_{t+1} (recall: $x_{t+1} \in \widetilde{f}(x_t, y_t)$)

- problem is *stationary* in that U and G do not depend on time

- solution is an infinite sequence $\{x_{t+1}\}_{t=0}^{\infty}$

- idea of dynamic programming:
  transform the problem into one of finding a (time-invariant) *function* rather than an infinite *sequence*

Principle of Optimality

- suppose $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is the solution to A1 and that $V\left(x_0^*\right)$ is finite. Then:

$$\begin{aligned} V\left(x_0^*\right) &= \textstyle\sum_{t=0}^{\infty} \beta^t U\left(x_t^*, x_{t+1}^*\right) \\ &= U\left(x_0^*, x_1^*\right) + \beta\left[\textstyle\sum_{t=0}^{\infty} \beta^t U\left(x_{t+1}^*, x_{t+2}^*\right)\right] \\ &= U\left(x_0^*, x_1^*\right) + \beta V\left(x_1^*\right) \end{aligned}$$

- but then, maximizing $V\left(x_0\right)$ must be equal to maximizing $U\left(x_0, x_1\right) + \beta V\left(x_1\right)$

- moreover, this has to be true at every date t, because the continuation value $V\left(x_{t+1}\right)$ takes the same form as the original problem

- this suggests the following time-invariant reformulation of the problem:

$$V(x) = \max_{y \in G(x)} \left\{U(x, y) + \beta V(y)\right\} \quad \text{for all } x \in X$$

where the solution is a mapping from the current state x to the future state y

Recursive Formulation: the Bellman Equation

- thus, we have converted A1 into A2 (Bellman Equation):

$$V(x_t) = \max_{x_{t+1}=\pi(x_t)} \{U(x_t, x_{t+1}) + \beta V(x_{t+1})\} \quad \text{for all } x \in X$$

  where x_t is the vector state and x_{t+1} is the vector of controls (tomorrow state)

- IMPORTANT NOTES:

  1. the infinite horizon plan is reduced to a two-period problem (today + continuation value)

  2. instead of finding an infinite sequence of controls, we aim at finding a time-invariant *policy function* that gives (implicitly) the optimal control:

  $$x_{t+1} = \pi(x_t)$$

  3. A2 is *recursive* in that $V(\cdot)$ appears both on the LHS and the RHS

  4. A2 is a functional equation, i.e., a function of a function

Recursive Formulation: Pros and Cons

- new problem: we need to find $V(\cdot)$

- once we have $V(\cdot)$, the policy function $x_{t+1} = \pi(x_t)$ can be found from:

$$V(x_t) \;=\; U(x_t, \pi(x_t)) + \beta V(\pi(x_t)) \quad \forall x_t \in X$$

- advantages of the recursive formulation:

  1. reduces the infinite horizon plan to a two-period problem (today and the future) $\rightarrow$ often gives better economic intuition

  2. some results guarantee existence and some properties of the solutions

  3. powerful numerical tools to find the solution

Additional Results

- Assumptions:

  1. $\lim_{n\to\infty}\sum_{n=0}^{\infty}\beta^n U\left(x_t^*, x_{t+1}^*\right)$ exists and is finite
  2. X is a compact subset of $\mathbb{R}^K$
  3. G is nonempty, compact valued, continuous and convex
  4. U is continuous, concave, differentiable and increasing in the state x

- then we can establish:

  1. equivalence of A1 (sequence problem) and A2 (recursive form)
  2. $V : X \to \mathbb{R}$ exists, is unique, bounded, continuous, concave, increasing and differentiable
  3. there exists a unique optimal plan with

  $$x_{t+1}^* = \pi(x_t^*)$$

  where $\pi : X \to X$ is a continuous policy function

Contraction Mappings

- contraction mappings are used to prove existence and uniqueness of V

- Definition:
  Let (S, d) be a metric space and $T : S \longrightarrow S$ be an operator mapping S into itself. If for some $\beta \in (0, 1)$

  $$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \quad \text{for all } z_1, z_2 \in S$$

  then T is a *contraction mapping* (with modulus β)

- a contraction mapping brings elements of S uniformly closer to each other

- Contraction Mapping Theorem:
  Let (S, d) be a complete metric space and $T : S \longrightarrow S$ a contraction. Then T has a unique fixed point, i.e. there exists a unique $\widehat{z} \in S$ such that:

  $$T\widehat{z} = \widehat{z}$$

Blackwell's Sufficient Conditions

- Let $X \subseteq \mathbb{R}^k$ and $B(X)$ be a space of bounded, real-valued functions $f : X \to \mathbb{R}$. Let $T : B(X) \to B(X)$ be an operator satisfying:
  1. (monotonicity) for any functions $f, g \in B(X)$, $f(x) \leq g(x)$ for all x implies $Tf(x) \leq Tg(x)$ for all x
  2. (discounting) there exists a $\beta \in (0,1)$ such that for all $f \in B(X)$, $c \geq 0$ and $x \in X$

  $$T(f(x) + c) \leq Tf(x) + \beta c$$

  Then, T is a contraction.

- Blackwell's sufficient conditions allow to verify that, with positive discounting, the Bellman equation is usually a contraction mapping

Dynamic Programming in Practice: FOCs

- Bellman equation:

$$V\left(x_t\right) = \max_{x_{t+1}=\pi(x_t)} \left\{U\left(x_t, x_{t+1}\right) + \beta V\left(x_{t+1}\right)\right\} \quad \text{for all } x \in X$$

- by the above results, the maximization is strictly concave and the maximand is differentiable
  $\longrightarrow$ for interior solutions, first order conditions are necessary and sufficient

- dynamic FOCs are called Euler equations:

$$D_{x_{t+1}} U\left(x_t, x^*_{t+1}\right) + \beta D V\left(x^*_{t+1}\right) = 0$$

  where D denotes the gradient

Envelope Theorem

- how do we evaluate $DV\left(x^*_{t+1}\right)$?

- from $V(x_t) = \max\{U(x_t, x_{t+1}) + \beta V(x_{t+1})\}$:

$$\begin{aligned} DV(x_t) &= D_{x_t}U\left(x_t, x^*_{t+1}\right) + D_{x_{t+1}}U\left(x_t, x^*_{t+1}\right)\frac{dx_{t+1}}{dx_t} + \beta DV(x^*_{t+1})\frac{dx_{t+1}}{dx_t} \\ &= D_{x_t}U\left(x_t, x^*_{t+1}\right) + \left[D_{x_{t+1}}U\left(x_t, x^*_{t+1}\right) + \beta DV(x^*_{t+1})\right]\frac{dx_{t+1}}{dx_t} \\ &= D_{x_t}U\left(x_t, x^*_{t+1}\right) \end{aligned}$$

  because the Euler equation requires the term $[\cdot] = 0$.

- intuition: $V(x_t)$ is maximized w.r.t. to $x_{t+1} \rightarrow$ small changes in x_{t+1} do not affect it (envelope theorem)

- since $x^*_{t+1} = \pi(x_t)$ and $x^*_{t+2} = \pi(x_{t+1})$:

$$DV\left(x^*_{t+1}\right) = D_{\pi(x_t)}U(\pi(x_t), \pi(\pi(x_t)))$$

Euler Equation

- back into the Euler equation:

$$D_{\pi(x)}U\left(x, \pi(x)\right) + \beta D_{\pi(x)}U\left(\pi(x), \pi(\pi(x))\right) = 0$$

  $\longrightarrow$ a functional equation in the unknown function $\pi(x)$

- IMPORTANT: solution must also satisfy the *transversality condition*

$$\lim_{t\to\infty} \beta^t D_x U\left(x_t^*, x_{t+1}^*\right) \cdot x_t^* = 0$$

  loosely: the discounted value of the stock x_t^* must be zero at infinity

- infinite-horizon equivalent of the terminal condition that all wealth should be consumed by the end of time

One-Dimentional Euler Equation

- suppose now that x and y are real numbers. The Euler eq. becomes

$$-\frac{\partial U\left(x_t, x_{t+1}\right)}{\partial x_{t+1}} = \beta V'\left(x_{t+1}\right)$$

intuitive: today's cost of increasing x_{t+1} (capital tomorrow?) has to be equal to its discounted marginal gain on future utility (indifference condition)

- using the Envelope Theorem:

$$\begin{aligned} \frac{\partial U\left(x, \pi(x)\right)}{\partial \pi(x)} + \beta\frac{\partial U\left(\pi(x), \pi(\pi(x))\right)}{\partial \pi(x)} &= 0 \\ \frac{\partial U\left(x_t, x_{t+1}\right)}{\partial x_{t+1}} + \beta\frac{\partial U\left(x_{t+1}, x_{t+2}\right)}{\partial x_{t+1}} &= 0 \end{aligned}$$

1. trade-off between today and tomorrow only
(effect of future continuation value second order by envelope theorem)
2. today's marginal cost of x_{t+1} = discounted marginal value tomorrow

Deriving the Transversality Condition

- suppose the last period is $t = T$. In the last period we have:

$$\max \beta^T U\left(x_T, x_{T+1}\right)$$

with FOC:

$$\beta^T \frac{\partial U\left(x_T, x_{T+1}\right)}{\partial x_{T+1}} x_{T+1} = 0$$

that is, either an interior x_{T+1} is optimal $(\partial U / \partial x_{T+1} = 0)$ or we go to the corner $x_{T+1} = 0$

- when $T = \infty$ the terminal condition becomes:

$$\lim_{T \to \infty} \beta^T \frac{\partial U\left(x_T, x_{T+1}\right)}{\partial x_{T+1}} x_{T+1} = 0$$

from the Euler eq. $\frac{\partial U\left(x_T, x_{T+1}\right)}{\partial x_{T+1}} = -\beta \frac{\partial U\left(x_{T+1}, x_{T+2}\right)}{\partial x_{T+1}}$ (note: now there is always a $T+1$!)

$$-\lim_{T \to \infty} \beta^{T+1} \frac{\partial U\left(x_{T+1}, x_{T+2}\right)}{\partial x_{T+1}} x_{T+1} = 0 \Longleftrightarrow \lim_{t \to \infty} \beta^t \frac{\partial U\left(x_t, x_{t+1}\right)}{\partial x_t} x_t = 0$$

How to Solve Functional Equations

- the Bellman equation is a functional equations

- no general way to solve it

- two approaches:

  1. guess on a functional form (often same form as U) and verify

  2. value function iterations: start from an arbitrary $V(x_t)$ and update it according to:

$$V^{new}(x_t) = \max\left\{U(x_t, x_{t+1}) + \beta V^{old}(x_{t+1})\right\}$$

  until it converges: $V^{new}(\cdot) = V^{old}(\cdot)$

- the contraction mapping theorem implies that iterations will converge
  $\rightarrow$ (2) is a powerful numerical method for finding the solution

Example 1: Optimal Growth

- consider the problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$
$$st \quad : \quad k_{t+1} = k_t^{\alpha} - c_t, \qquad k_0$$

- in this case we have:
  $x_t = k_t$ (state), $x_{t+1} = k_{t+1}$ (control) and $c_t = k_t^{\alpha} - k_{t+1}$

- Bellman equation:

$$V(x_t) = \max_{x_{+1}} \{\ln(x_t^{\alpha} - x_{+1}) + \beta V(x_{+1})\}$$

  or, to highly it is a functional eq:

$$V(x) = \ln(x^{\alpha} - \pi(x)) + \beta V(\pi(x))$$

Example 1: First Order Conditions

- recall:

$$U\left(x,\pi(x)\right)=\mathsf{ln}\left(x^{\alpha}-\pi\left(x\right)\right)$$

Euler equation:

$$\frac{\partial U\left(x,\pi(x)\right)}{\partial\pi(x)}+\beta\frac{\partial U\left(\pi(x),\pi(\pi(x))\right)}{\partial\pi(x)}=0$$

thus:

$$\frac{1}{x^{\alpha}-\pi\left(x\right)}=\beta\frac{\partial\,\mathsf{ln}\left(\pi\left(x\right)^{\alpha}-\pi\left(\pi\left(x\right)\right)\right)}{\partial\pi\left(x\right)}=\frac{\beta\alpha\pi\left(x\right)^{\alpha-1}}{\pi\left(x\right)^{\alpha}-\pi\left(\pi\left(x\right)\right)}$$

- no general way of solving functional equations

- guess-and-verify often works

Example 1: Guess-and-Verify

- conjecture: $\pi(x) = ax^\alpha \rightarrow \pi(\pi(x)) = a(ax^\alpha)^\alpha$
  where a is a constant to be found

- substitute into the functional equation:
$$\frac{1}{x^\alpha - ax^\alpha} = \beta \frac{\alpha a^{\alpha-1} x^{\alpha(\alpha-1)}}{a^\alpha x^{\alpha^2} - a^{1+\alpha} x^{\alpha^2}} = \frac{\alpha\beta}{a} \frac{1}{x^\alpha - ax^\alpha}$$
  satisfied for $a = \alpha\beta$

- thus:
$$\begin{aligned} k_{t+1} &= \alpha\beta k_t^\alpha \\ c_t &= k_t^\alpha - k_{t+1} = (1 - \alpha\beta) k_t^\alpha \end{aligned}$$
  as in a discrete-time Solow model!

- transversality is satisfied because k_t converges to a finite level

Example 2: Optimal Saving

- infinitely lived agent with initial assets a_0 facing a wage w and interest rate r:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$st \quad : \quad a_{t+1} = (1+r) a_t + w - c_t, \quad a_0$$

- debt limit: $a_t \geq -\left(\frac{w}{r} + a_0\right)$ (can never borrow more than total present value of wealth)

- substitute out c_t from the b.c. $c_t = (1+r) a_t + w - a_{t+1}$

- Bellman equation:

$$V(a_t) = \max_{a_{t+1}} \{u((1+r) a_t + w - a_{t+1}) + V(a_{t+1})\}$$

Example 2: Consumption Euler Equation

- recall $u(a_t, a_{t+1}) = u((1+r)a_t + w - a_{t+1})$
  using $\frac{\partial u(a_t, a_{t+1})}{\partial a_{t+1}} + \beta \frac{\partial u(a_{t+1}, a_{t+2})}{\partial a_{t+1}} = 0$:

$$u'(c_t) = \beta (1+r) u'(c_{t+1})$$

- *Euler equation for consumption growth:*
  indifference between consuming the marginal unit today or consuming it tomorrow
  if $r = \beta^{-1} - 1 \longrightarrow$ flat consumption profile
  if $r > \beta^{-1} - 1 \longrightarrow$ declining $u'(\cdot) \longrightarrow c_t$ increases between t and $t+1$
  if $r < \beta^{-1} - 1 \longrightarrow$ increasing $u'(\cdot) \longrightarrow c_t$ declines between t and $t+1$

- important remarks:
  1. identifies the *slope* of the consumption path, to find the *level* we need to use the budget constraint and the transversality condition
  2. the slope of consumption path is independent of income

Summary

- we have studied stationary dynamic programming problems where

  1. planning horizon is infinite

  2. both the payoff and the constraint functions are time independent

- we have find conditions (Euler eq. + transversality) that identify optimal solution

- easy to show how these conditions apply to finite horizon case

- under some additional conditions, the Euler equation and the transversality condition are valid also for nonstationary problems

Advanced Macro - Lecture 4

Dynamic Optimization in Continuous Time

Ref: Acemoglu ch. 7

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Canonical Problem in Continuous Time

- can be written as:

$$\max_{y_t} W = \int_0^{t_1} f(t, x_t, y_t)\, dt$$
$$st \quad : \quad \dot{x}_t = g(t, x_t, y_t)$$
$$: \quad y_t \in Y, \; x_t \in X \quad x_0 \text{ given}$$

$x_t \in X \subset \mathbb{R}^{K_x}$: vector of state variables (predetermined)
$y_t \in Y \subset \mathbb{R}^{K_y}$: vector of control variables (choice variables)
$f : \mathbb{R} \times X \times Y \to \mathbb{R}$ objective function
$g(x_t)$ constraint function $=$ law of motion of the state

- t_1 might be ∞ (infinite horizon)

- note: discounting is included in $f(t)$ - more general formulation

Continuous Time: New Issues

- new difficulties:

  1. even with finite horizon, we are choosing a function $y : [0, t_1] \to Y$ rather than a vector or a finite dimensional object

  2. the constraint is a differential equation

- to make progress, we start with the finite-horizon, one dimensional case

Finite-Horizon, One-Dimensional Problem

- assume (throughout) that:
  f and g are continuously differentiable functions
  Y is nonempty and convex

- suppose that:
  W is finite for any admissible x_t and y_t
  $\exists$ a continuous function y_t^* defined over $[0, t_1]$ with $y_t^* \in \text{Int}Y$ that achieves the optimum

- thus, the solution is continuous and interior

- variational argument:
  - as in standard calculus, there should be no gain from small variations from y_t^*

- problem: an admissible variation here is a change in an entire *function*

Admissible Variations

- consider the following *variation* from the optimal plan:

$$y_{t,\varepsilon} = y_t^* + \varepsilon\eta_t$$

  where η_t is an arbitrary fixed continuous function and $\varepsilon \in \mathbb{R}$

- note: given η_t, by varying ε, we obtain different sequences of controls $y_{t,\varepsilon}$ (a family of functions)

- not all variations will be feasible: $y_{t,\varepsilon}$ may be outside Y for some ε

- yet, since $y_t^* \in$Int$Y$ and a continuous function over a compact set $[0, t_1]$ is bounded, we can always find $\varepsilon_\eta > 0$ such that for any η_t we have:

$$y_t^* + \varepsilon\eta_t \in \text{Int}Y$$

  for all $\varepsilon < \varepsilon_\eta$

- thus, $y_t^* + \varepsilon\eta_t$ is an admissible variation for small ε

Variational Argument I

- define $x_{t,\varepsilon}$ as the path of states generated by the path of controls $y_{t,\varepsilon}$ and $W_\varepsilon = \int_0^{t_1} f\left(t, x_{t,\varepsilon}, y_{t,\varepsilon}\right) dt$

- by the fact that $\hat{y}_t$ is optimal, we must have $W_\varepsilon \leq W_{\varepsilon=0}$

- in analogy to the Lagrangian, use the constraint function to write:

$$W_\varepsilon = \int_0^{t_1} \left\{ f\left(t, x_{t,\varepsilon}, y_{t,\varepsilon}\right) + \lambda_t \left[g\left(t, x_{t,\varepsilon}, y_{t,\varepsilon}\right) - \dot{x}_{t,\varepsilon} \right] \right\} dt$$

  for any function $\lambda_t : [0, t_1] \to \mathbb{R}$ (will be the costate variable)

- note I: we are just adding $\lambda_t [\cdot] = 0$ to the objective function

- note II: optimality requires $W'_{\varepsilon=0} = 0$ (W cannot be improved by changing $\hat{y}_t$ slightly)

Variational Argument II

- to get rid of $\dot{x}_{t,\varepsilon}$, take $\int_0^{t_1} \lambda_t \dot{x}_{t,\varepsilon} dt$ and integrate by parts:

$$\int_0^{t_1} \lambda_t \dot{x}_{t,\varepsilon} dt = \left[\lambda_t x_{t,\varepsilon}\right]_0^{t_1} - \int_0^{t_1} \dot{\lambda}_t x_{t,\varepsilon} dt = \lambda_{t_1} x_{t_1,\varepsilon} - \lambda_0 x_0 - \int_0^{t_1} \dot{\lambda}_t x_{t,\varepsilon} dt$$

- back into W_ε:

$$W_\varepsilon = \int_0^{t_1} \left\{ f\left(t, x_{t,\varepsilon}, y_{t,\varepsilon}\right) + \lambda_t g\left(t, x_{t,\varepsilon}, y_{t,\varepsilon}\right) + \dot{\lambda}_t x_{t,\varepsilon} \right\} dt - \lambda_{t_1} x_{t_1,\varepsilon} + \lambda_0 x_0$$

- differentiate with respect to ε :

$$\begin{aligned} W'_\varepsilon &= \int_0^{t_1} \left\{ f_x x_\varepsilon + f_y \eta_t + \lambda_t g_x x_\varepsilon + \lambda_t g_y \eta_t + \dot{\lambda}_t x_\varepsilon \right\} dt - \lambda_{t_1} \frac{\partial x_{t_1,\varepsilon}}{\partial \varepsilon} \\ &= \int_0^{t_1} \left\{ f_x + \lambda_t g_x + \dot{\lambda}_t \right\} x_\varepsilon dt + \int_0^{t_1} \left\{ f_y + \lambda_t g_y \right\} \eta_t dt - \lambda_{t_1} \frac{\partial x_{t_1,\varepsilon}}{\partial \varepsilon} \end{aligned}$$

where f_x f_y x_ε now indicate *derivatives*

Necessary Conditions

- optimality requires

  $$W'_{\varepsilon=0} = \int_0^{t_1} \left\{ f_x + \lambda_t g_x + \dot{\lambda}_t \right\} x_\varepsilon dt + \int_0^{t_1} \{ f_y + \lambda_t g_y \} \eta_t dt - \lambda_{t_1} \frac{\partial x_{t_1,\varepsilon}}{\partial \varepsilon} = 0$$

  for *any* η_t: this is possible if all three terms of $W'_{\varepsilon=0}$ are zero

- for the first term to be zero we need

  $$\dot{\lambda}_t = -\left[f_x + \lambda_t g_x \right]$$

- for the second and third we need

  $$\begin{aligned} f_y + \lambda_t g_y &= 0 \\ \lambda_{t_1} &= 0 \end{aligned}$$

Simplified Maximum Principle

- if we define the Hamiltonian as:

$$H(t, x, y, \lambda) \equiv f(t, x_t, y_t) + \lambda_t g(t, x_t, y_t)$$

then, the necessary conditions are:

$$\begin{aligned} H_y &= 0 \\ \dot{\lambda}_t &= -H_x \quad \text{and} \quad \lambda_{t_1} = 0 \\ \dot{x}_t &= H_\lambda \quad \text{and} \quad x_0 \text{ given} \end{aligned}$$

1. the solution is given by a system of two differential eq. and two boundary conditions

2. the condition $\dot{x}_t = H_\lambda$ simply requires the constraint to hold

3. in analogy with Lagrangians, we find a set of "multipliers" λ_t (costate) that give the value of a marginal change in x_t (to be verified later)

4. thus, $\lambda_{t_1} = 0$ means that, after the planning horizon, there should be no value to having more x

Economic Interpretation of the Hamiltonian

- x_t and y_t affect W through two channels:

  1. the direct effect through $f(t, x_t, y_t)$

  2. the effect through the change in the state $\dot{x}_t$, with value

$$\lambda_t \dot{x}_t = \lambda_t g(t, x_t, y_t)$$

- the Hamiltonian is the sum the two effects:

$$H(t, x, y, \lambda) \equiv f(t, x_t, y_t) + \lambda_t g(t, x_t, y_t)$$

- $H_y = 0 \rightarrow$ as in dynamic programming, the control y is chosen so as to maximize the sum of current and future payoffs

Current Value Hamiltonian

- we often have problems where t only enters as exponential discounting:

$$\max_{y_t} W = \int_0^{t_1} e^{-\rho t} f(x_t, y_t)\, dt, \quad \rho > 0$$

- the Hamiltonian is:

$$\begin{aligned} H &\equiv e^{-\rho t} f(x_t, y_t) + \lambda_t g(t, x_t, y_t) \\ &= e^{-\rho t} [f(x_t, y_t) + \mu_t g(t, x_t, y_t)] = e^{-\rho t} \hat{H} \end{aligned}$$

  where $\mu_t = e^{\rho t} \lambda_t$ and $\hat{H}$ is the *current value* Hamiltonian

- then, the necessary conditions can be written as:

$$\begin{aligned} \hat{H}_y &= 0 \quad \text{with} \quad \hat{H} = f + \mu_t g \\ \rho\mu_t - \dot{\mu}_t &= \hat{H}_x \quad \text{and} \quad e^{-\rho t_1}\mu_{t_1} = 0 \\ \dot{x}_t &= \hat{H}_\mu \quad \text{and} \quad x_0 \text{ given} \end{aligned}$$

- now, μ_t is the *current* value of x_t

Alternative Derivation with Exponential Discounting

- use the logic of dynamic programming in discrete time and let the time period dt go to zero

- Bellman equation:

$$V(x_t) = \max_{y_t} \{f(x_t, y_t)\, dt + (1 - \rho dt)\, V(x_t + g(x_t, y_t)\, dt)\}$$

  where we used the approximation $e^{-\rho dt} \simeq 1 - \rho dt$.

- denote the (shadow) value of x_t by $\mu_t \equiv V'(x_t)$. Then, the FOC is:

$$f_y dt + (1 - \rho dt)\, \mu_{t+dt} g_y dt = 0$$

  dividing by dt and letting $dt \longrightarrow 0$:

$$f_y + \mu_t g_y = 0$$

Alternative Derivation Continued

- next, use the Bellman eq. to find the formula for μ_t :

$$
\begin{aligned}
V'(x_t) &= f_x dt + (1-\rho dt)\,\mu_{t+dt}\,(1+g_x dt) \equiv \mu_t \\
0 &= f_x dt + \mu_{t+dt} - \mu_t - \rho dt \mu_{t+dt} + (1-\rho dt)\,\mu_{t+dt} g_x dt
\end{aligned}
$$

- dividing by dt and letting $dt \to 0$:

$$
\begin{aligned}
\lim_{dt \to 0} \left[f_x + \frac{\mu_{t+dt} - \mu_t}{dt} - \rho \mu_{t+dt} + (1-\rho dt)\,\mu_{t+dt} g_x \right] &= 0 \\
f_x + \dot{\mu}_t - \rho \mu_t + \mu_t g_x &= 0
\end{aligned}
$$

- this confirms our claim that μ_t is the shadow value of x_t

The Hamilton-Jacobi-Bellman Equation

- from the Bellman equation:

$$\begin{aligned} V(x_t) &= f(x_t, y_t)\,dt + (1 - \rho dt)\,V(x_{t+dt}) \\ \frac{V(x_t) - V(x_{t+dt})}{dt} &= f(x_t, y_t) - \rho V(x_{t+dt}) \end{aligned}$$

  letting $dt \to 0$

$$\begin{aligned} -\dot{V}(x_t) &= f(x_t, y_t) - \rho V(x_t) \\ f(x_t, y_t) + \dot{V}(x_t) &= \rho V(x_t) \end{aligned}$$

- intuition:
  think of $V(x_t)$ as the value of an asset
  return from keeping the asset $=$ dividend it pays $f(x_t, y_t)$ plus the capital gain/loss $\dot{V}(x_t)$
  cost of keeping the asset ρ ($=$ rate of return on alternative asset, consumption)
  optimality requires the maximized value $V(x_t)$ and its rate of change to be consistent with this no-arbitrage condition

Sufficient Conditions

- necessary conditions for optimality are also sufficient when one of the following conditions holds:

  1. (Mangasarian) the Hamiltonian

     $$H(t, x, y, \lambda) \equiv f(t, x_t, y_t) + \lambda_t g(t, x_t, y_t)$$

     is concave in x and y. Thus, sufficiency is guarantees when both f and g are concave in x and y.

  2. (Arrow) the *maximized* Hamiltonian H^* (that is, H when y is chosen optimally) is concave in x

- note: condition (1) implies (2). Thus, (1) is more restrictive. However, (2) is more difficult to check

Infinite-Horizon Problem

- previous conditions extend to the infinte-horizon case, except for the transversality condition

- in analogy with the discrete-time case, the transversality condition for the *discounted infinite horizon optimal control* is:

$$\lim_{t \to \infty} \lambda_t x_t = 0$$

or

$$\lim_{t \to \infty} e^{-\rho t} \mu_t x_t = 0$$

for the current value costate $\mu_t = e^{\rho t} \lambda_t$

- note: in the general case without exponential discounting, a milder transversality condition may apply:

$$\lim_{t \to \infty} H(t, x, y, \lambda) = 0$$

Optimal Control with Discounting in Practice: Summary

1. build the current-value Hamiltonian

$$\hat{H} = f(x_t, y_t) + \mu_t g(t, x_t, y_t)$$

   where $f(x_t, y_t)$ is the instantaneous payoff function and $g(t, x_t, y_t) = \dot{x}_t$

2. find a candidate solution that satisfies:

$$\begin{aligned} \hat{H}_y &= 0 \\ \hat{H}_x &= \rho\mu_t - \dot{\mu}_t \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\mu_t x_t = 0 \\ \dot{x}_t &= g(t, x_t, y_t) \quad \text{and} \quad x_0 \text{ given} \end{aligned}$$

3. verify the concavity condition

Advanced Macro - Lecture 5

The Neoclassical Growth Model

Ref: Acemoglu ch. 8, Barro and Sala-i-Martin ch. 2

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The Neoclassical Growth Model

- also known as Ramsey-Cass-Koopmans model

- differences from the Solow model: endogenous saving decisions

- it is the core of many macroeconomic models

- infinite horizon, continuous time

Technology

- no technological progress (for now)

- production possibility set = neoclassical production function (CRS, Inada conditions):

$$
\begin{aligned}
Y_t &= F(K_t, L_t) \\
y_t &= \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t) \quad \text{(intensive form)}
\end{aligned}
$$

- factor and product markets are competitive
  $\longrightarrow$ profit maximization yields:

$$
\begin{aligned}
\text{rental rate:} \qquad & R_t = F_K = f'(k_t) \\
\text{wage:} \qquad & w_t = F_L = f(k_t) - k_t f'(k_t)
\end{aligned}
$$

Preferences

- population grows at rate n: $L_t = e^{nt} L_0$ and $L_0 = 1$
  population $=$ workforce (inelastic labor supply)

- objective function of the representative household:

$$U_0 = \int_0^\infty e^{(n-\rho)t} u(c_t)\, dt$$

  $\rho > n$ to ensure U_0 is finite (stronger condition with technological progress)
  $c_t = C_t / L_t$ is consumption per capita $\rightarrow$ total utility $= u(c_t) L_t = e^{nt} u(c_t)$
  $u(c_t)$ is increasing, concave and twice differentiable

- note: full altruism implicitly assumed

The Flow Budget Constraint

- define $a_t = A_t / L_t$ as assets per capita

$$\begin{aligned} \dot{a}_t &= (r_t - n)\, a_t + w_t - c_t \\ a_0 &= \text{given} \end{aligned}$$

- yet, the flow budget constraint does not contain all the relevant constraints

- define the *average interest rate* between 0 and t:

$$\bar{r}_t = \frac{1}{t} \int_0^t r_s ds$$

$\rightarrow$ the conversion factor between date t and 0 is $e^{-\bar{r}_t t}$ (present value factor)

The Lifetime Budget Constraint

- solve the flow b.c. as a linear differential equation:

$$\dot{a}_t - (r_t - n)\, a_t = w_t - c_t$$

multiply by the factor $e^{-(\bar{r}_t - n)t}$ and integrate

$$\begin{aligned} \int \left[\dot{a}_t - (r_t - n)\, a_t\right] e^{-(\bar{r}_t - n)t} dt &= \int (w_t - c_t)\, e^{-(\bar{r}_t - n)t} dt \\ a_t e^{-(\bar{r}_t - n)t} + b &= \int (w_t - c_t)\, e^{-(\bar{r}_t - n)t} dt \end{aligned}$$

- evaluating at $t = 0$ and using the initial condition we get

$$b = -a_0$$

No Ponzi-Game Condition

- over the lifetime $[0, T]$ the consolidated budget constraint is:

$$\int_0^T c_t e^{-(\bar{r}_t - n)t} dt + a_T e^{-(\bar{r}_T - n)T} \leq \int_0^T w_t e^{-(\bar{r}_t - n)t} dt + a_0$$

  $\rightarrow$ discounted sum of consumption plus end of the period assets (total expenditures) = to initial income plus the discounted value of all labor income

- at the end of time, nobody can borrow $\rightarrow a_T e^{-(\bar{r}_T - n)T} \geq 0$. In infinite horizon:

$$\lim_{t \to \infty} e^{-(\bar{r}_t - n)t} a_t \geq 0$$

  otherwise, a consumer will simply borrow forever!

- these conditions are NOT guaranteed by the flow budget constraint

- thus, we must add the no-Ponzi condition as an additional constraint

Household Problem

- in sum:

$$\begin{aligned} \max U_0 &= \int_0^\infty e^{-(\rho-n)t} u(c_t)\, dt \\ st &: \dot{a}_t = (r_t - n)\, a_t + w_t - c_t \\ a_0 &= \text{given}, \qquad \lim_{t\to\infty} e^{-(\bar{r}_t - n)t} a_t \geq 0 \end{aligned}$$

- current value Hamiltonian:

$$\hat{H} = u(c_t) + \mu_t \left[(r_t - n)\, a_t + w_t - c_t \right]$$

necessary conditions:

$$\begin{aligned} \hat{H}_c &= u'(c_t) - \mu_t = 0 \\ \hat{H}_a &= \mu_t (r_t - n) = -\dot{\mu}_t + (\rho - n)\, \mu_t \\ 0 &= \lim_{t\to\infty} e^{-(\rho-n)t} \mu_t a_t \end{aligned}$$

Necessary Conditions

- rearranging the condition for $\hat{H}_a$:

$$\frac{\dot{\mu}_t}{\mu_t} = \rho - r_t \tag{2}$$

- from the condition for $\hat{H}_c$:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \ln u'(c_t) &= \ln \mu_t \end{aligned}$$

time-differentiate and use (2):

$$\frac{u''(c_t)\, c_t}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t} = \rho - r_t$$

- thus:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta_t}(r_t - \rho)$$

where $\theta_t = -\frac{u''(c_t)c_t}{u'(c_t)}$ is the elasticity of marginal utility

Euler Equation: Consumption Growth and the Interest Rate

- concavity implies that agents prefer a smooth consumption profile:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta_t}(r_t - \rho)$$

- consumption grows over time when the discount factor is less than the interest rate

- the speed is higher when is θ_t low

- in other words $1/\theta_t$ captures the *intertemporal elasticity of substitution*

- equivalently, θ_t captures the desire for *consumption smoothing* (eg, $\theta_t \to \infty \to \frac{\dot{c}_t}{c_t} \to 0 \to$ flat consumption profile)

Euler Equation: Alternative Interpretation

- rearrange the Euler equation as:

$$r_t = \theta_t \frac{\dot{c}_t}{c_t} + \rho$$

- in a growing economy, consumption "tomorrow" is less valuable for two reasons:
  1. because the future is discounted at rate ρ
  2. because marginal utility tomorrow is lower as long as $\frac{\dot{c}_t}{c_t} > 0$ (by how much? depends on curvature θ_t)

- the Euler equation says that the interest rate must be just enough to compensate these two effects
  $\rightarrow$ indifference at the margin between consuming and saving

The Consumption Function

- if θ is constant, integrating the Euler equation:

$$c_t = c_0 \exp\left(\frac{\bar{r}_t - \rho}{\theta} t\right)$$

- where do we find c_0? From the lifetime budget constraint:

$$\int_0^\infty c_t e^{-(\bar{r}_t - n)t} dt = \int_0^\infty w_t e^{-(\bar{r}_t - n)t} dt + a_0$$

substituting c_t we obtain:

$$c_0 = \frac{\int_0^\infty w_t e^{-(\bar{r}_t - n)t} dt + a_0}{\int_0^\infty e^{\left(\bar{r}_t \frac{1-\theta}{\theta} - \frac{\rho}{\theta} + n\right)t} dt}$$

- note: if $\theta = 1$ (log utility): consumption is a constant fraction of lifetime wealth

Transversality Condition

- recall:

$$\frac{\dot{\mu}_t}{\mu_t} = \rho - r_t \to \mu_t = \mu_0 e^{(\rho - \bar{r}_t)t}$$

- substitute into the transversality condition:

$$0 = \lim_{t\to\infty} e^{-(\rho-n)t} \mu_0 e^{(\rho-\bar{r}_t)t} a_t = \lim_{t\to\infty} e^{-(\bar{r}_t - n)t} \mu_0 a_t = 0$$

- implying that the no-Ponzi condition must hold with equality

$$\lim_{t\to\infty} e^{-(\bar{r}_t - n)t} a_t = 0$$

  because $\mu_0 = u'(c_0)$ is positive

- intuitive: we cannot borrow more than our lifetime income, but we do not want to leave resources unspent either!

General Equilibrium

- in aggregate, per capita assets must be equal to per capita capital:

$$a_t = k_t$$

- prices must be consistent with profit maximization:

$$\begin{aligned} r_t &= R_t - \rho = f'(k_t) - \delta \\ w_t &= f(k_t) - f'(k_t)\, k_t = f(k_t) - (r_t + \delta)\, k_t \end{aligned}$$

- thus, substituting into the law of motion of a_t :

$$\begin{aligned} \dot{a}_t &= (r_t - n)\, a_t + w_t - c_t \\ \dot{k}_t &= f(k_t) - c_t - (n + \delta)\, k_t \end{aligned}$$

  we get the aggregate resource constraint of the economy

Equilibrium Dynamics

- in sum, the equilibrium dynamics are characterized by the system:

$$\begin{aligned} \dot{k}_t &= f(k_t) - c_t - (n+\delta) k_t \\ \frac{\dot{c}_t}{c_t} &= \frac{1}{\theta_t}\Big(f'(k_t) - \delta - \rho\Big) \end{aligned}$$

plus an initial condition k_0 and a boundary condition, $\lim_{t\to\infty} e^{-(\bar{r}_t - n)t} k_t = 0$

- this is a system of two non-linear differential equations. The system has a steady-state when $\dot{k}_t = \dot{c}_t = 0$:

$$\begin{aligned} \dot{k}_t &= 0 = f(k_t) - c_t - (n+\delta) k_t \rightarrow c = f(k) - (n+\delta) k \\ \dot{c}_t &= 0 = \frac{1}{\theta_t}\Big(f'(k_t) - \delta - \rho\Big) \rightarrow f'(k) = \delta + \rho \end{aligned}$$

- will the economy converge to this steady-state?
yes and we can show this drawing the phase diagram

Phase Diagram I

1. draw the loci $\dot{k}_t = 0$ and $\dot{c}_t = 0$ in the space (k, c)
   the steady-state is the crossing point

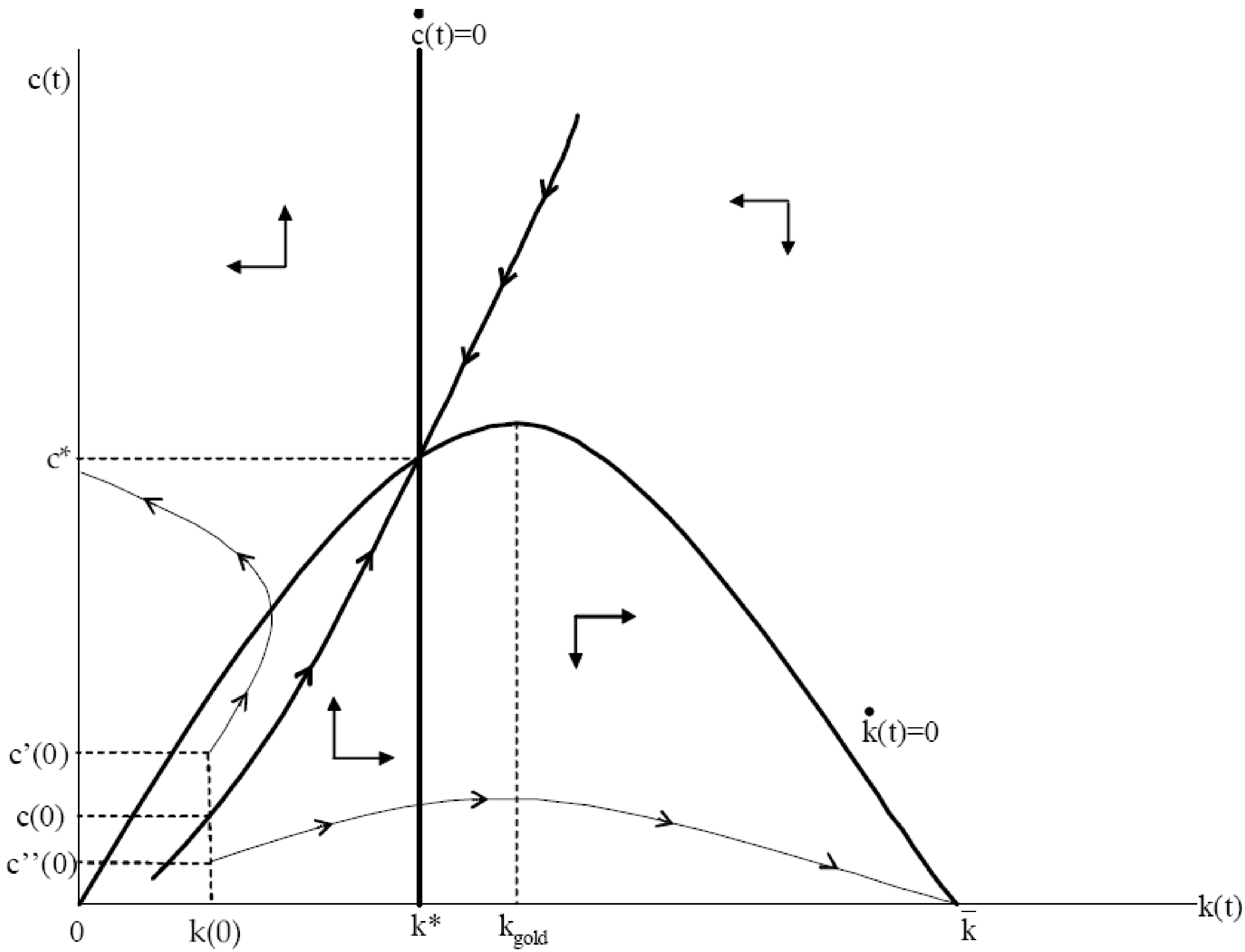
2. use the differential equations to study the qualitative dynamics out of steady-state

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &> 0 \text{ when } f'(k) > \delta + \rho \\ \dot{k}_t &> 0 \text{ when } f(k_t) - (n+\delta) k_t > c_t \end{aligned}$$

3. inspection of the phase diagram show that, for any k_0, there is a unique path leading to the steady state (recall that trajectories can never intersect!)
   $\longrightarrow$ the steady-state is saddle-path stable: for any k_0 there is a unique c_0 that sets the economy on the stable path

4. prove that this trajectory is the unique equilibrium by showing that other trajectories eventually violate some necessary condition

Phase Diagram II



Phase Diagram III

- suppose c_0 is *above* the saddle-path
  $\rightarrow$ k_t will reach zero in finite time
  $\rightarrow$ but $k_t = 0 \rightarrow c_t = 0$ and this require a jump of consumption that violates the Euler equation

- suppose c_0 is *below* the saddle-path
  $\rightarrow$ k_t will eventually cross the golden rule level $\left(r_{gold} = f'\left(k_{gold}\right) - \delta = n\right)$
  $\rightarrow$ but then $r_t < n$ implies a violation of the transversality condition:
  $\lim_{t\rightarrow\infty} e^{-(\bar{r}_t - n)t}\overline{k} > 0$

- on the contrary, the steady-state satisfies the transversality
  $\lim_{t\rightarrow\infty} e^{-(\bar{r}_t - n)t}k^* > 0$
  because k^* is constant and $f'(k^*) - \delta = \rho > n$ by assumption

- note: $k^* < k_{gold} \rightarrow$ no "overaccumulation", yet c^* is not maximized
  why? because earlier consumption is preferred due to discounting
  $\rightarrow$ the golden rule is not optimal!

Linear Approximations (local dynamics)

- an alternative way to analyze the dynamic system and study stability is to linearize it around the steady-state. In matrix form:

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} f'(k^*) - n - \delta & -1 \\ \frac{c^*}{\theta} f''(k^*) & 0 \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix}$$

  In steady-state we have $f'(k) = \delta + \rho$, so that:

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \end{bmatrix} = \begin{bmatrix} \rho - n & -1 \\ \frac{c^*}{\theta} f''(k^*) & 0 \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix}$$

  this system is saddle-path stable when the number of negative (stable) eigenvalues λ is equal to the number of state variables.

- eigenvalues λ of the matrix of coefficients can be found imposing:

$$\det \begin{bmatrix} \rho - n - \lambda & -1 \\ \frac{c^*}{\theta} f''(k^*) & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - (\rho - n)\lambda + \frac{c^*}{\theta} f''(k^*) = 0$$

  since $\frac{c^*}{\theta} f''(k^*) < 0$ this second order polynomial has two roots, one positive, one negative $\rightarrow$ saddle path stability

Optimal Growth

- a social planner maximizes:

$$\max_{c_t} U_0 = \int_0^\infty e^{-(\rho - n)t} u(c_t)\, dt$$

subject to the aggregate resource constraint

$$\dot{k}_t = f(k_t) - c_t - (n + \delta) k_t$$

and $k_0, k_t \geq 0$

- in this economy, the Welfare theorems apply
$\longrightarrow$ we know that the Pareto optimum and the competitive equilibrium coincide

- you will verify this in Problem Set 2

Exogenous Technical Progress

- assume now that

$$Y_t = F(K_t, A_t L_t)$$

with $A_t = e^{gt} A_0 \rightarrow$ labor augmenting technical progress at exogenous rate g

- convenient to normalize variable as:

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = f\left(\hat{k}_t\right), \quad \hat{k}_t = \frac{K_t}{A_t L_t}$$

these normalized variables will be constant in steady-state

- law of motion of $\hat{k}_t$:

$$\dot{\hat{k}}_t = f\left(\hat{k}_t\right) - c_t - (n + \delta + g)\hat{k}_t$$

Exogenous Technical Progress: Equilibrium

- household problem as before: $\frac{\dot{c}_t}{c_t} = \frac{1}{\theta_t}(r_t - \rho)$
  however: $\hat{c}_t = c_t / A_t \rightarrow$ thus:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{\dot{c}_t}{c_t} - g = \frac{1}{\theta_t}(r_t - \rho - g\theta_t)$$

- in a steady-state: $\dot{\hat{k}}_t = \dot{\hat{c}}_t = 0$ and $r_t = f'\left(\hat{k}_t\right) - \delta = r$
  $\rightarrow \frac{\dot{c}_t}{c_t} = g = \frac{1}{\theta_t}(r - \rho)$ only possible if $\theta_t = \theta$
  thus, a *balanced growth path* requires preferences to be *isoelastic* (constant elasticity of marginal utility) at least asymptotically

- transversality condition:

$$\lim_{t \to \infty} e^{-(\bar{r}_t - n)t} k_t = 0 \rightarrow \lim_{t \to \infty} e^{-(\bar{r}_t - n)t} \hat{k}_t A_0 e^{gt} = 0$$

  thus, the relevant condition is $\bar{r}_t > n + g$ or $\rho > n + g(1 - \theta)$

Conclusions

- main contributions:
  1. deeper understanding of the determinants of capital accumulation
  2. can do welfare analysis
  3. dynamic model of consumption

- new insights on the sources of income differences:
  compared to the Solow model, not much...

- yet, it clarifies the nature of the economic decisions leading to growth

- this paves the way for further analysis

Advanced Macro - Lecture 6

Growth with Overlapping Generations

Ref: Acemoglu ch. 9

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Growth with Overlapping Generations

- in many situations, households do not have an infinite horizon

- when households have a finite planning horizon and new households are born over time
  $\longrightarrow$ the economy does not admit a representative household

- decisions made by older generations affect prices faced by younger generations

- OLG models allow us to study these new issues

- key new result: CE (Competitive Equilibrium) may be inefficient
  $\longrightarrow$ role for government intervention

Baseline OLG Model: Setup

- time t is discrete and runs to infinity

- each individual lives for two periods (young$_{(1)}$ $\longrightarrow$ old$_{(2)}$)

- every period new individuals are born
  - with exponential population growth the size of generation t is:

$$L_t = (1+n)^t \, L_0$$

- individuals can only work when young (first period of life)

Technology

- same as in the Ramsey model

- production possibility set = neoclassical production function (CRS, Inada conditions):

$$
\begin{aligned}
Y_t &= F(K_t, L_t) \\
y_t &= \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t) \quad \text{(intensive form)}
\end{aligned}
$$

- factor and product markets are competitive
  to simplify assume $\delta = 1$ (full depreciation)
  $\rightarrow$ profit maximization yields:

$$
\begin{aligned}
\text{rental rate:} \qquad & R_t = F_K = f'(k_t) = 1 + r_t \\
\text{wage:} \qquad & w_t = F_L = f(k_t) - k_t f'(k_t)
\end{aligned}
$$

Consumption Decisions

- denote s_t as savings of a single member of generation t. s_t is the solution to:

$$\begin{aligned} \max U_t &= u(c_{1t}) + \beta u(c_{2t+1}) \\ st &: c_{1t} + s_t \leq w_t \\ &: c_{2t+1} \leq s_t R_{t+1} \end{aligned}$$

- the Euler equation:

$$u'(c_{1t}) = \beta R_{t+1} u'(c_{2t+1})$$

together with the budget constraint implicitly define the saving function:

$$s_t = s(w_t, R_{t+1})$$

- all new savings are invested in capital:

$$K_{t+1} = s(w_t, R_{t+1}) L_t$$

Equilibrium Dynamics

- divide K_{t+1} by $L_{t+1} = (1+n)\,L_t$:

$$\frac{K_{t+1}}{L_{t+1}} = k_{t+1} = \frac{s\,(w_t, R_{t+1})}{L_{t+1}} L_t = \frac{s\,(w_t, R_{t+1})}{1+n}$$

- substitute w_t and R_{t+1} :

$$k_{t+1} = \frac{s\left(f\,(k_t) - k_t f'\,(k_t)\,,\;\; f'\,(k_{t+1})\right)}{1+n}$$

$\longrightarrow$ fundamental law of motion of the OLG economy (as in Solow)

- a steady-state (if exists) is a solution to:

$$k^* = \frac{s\left(f\,(k^*) - k^* f'\,(k^*)\,,\;\; f'\,(k^*)\right)}{1+n}$$

- problem: $s\,(w_t, R_{t+1})$ can take many forms $\longrightarrow$ potentially complex dynamics

Saving Function: CRRA Utility

- assume:

$$U_t = \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + \beta \frac{c_{2t+1}^{1-\theta} - 1}{1-\theta}, \quad \theta > 0$$

- Euler equation:

$$\frac{c_{2t+1}}{c_{1t}} = (\beta R_{t+1})^{1/\theta}$$

substitute c_{1t} and c_{2t+1} from the budget constraint:

$$\frac{s_t R_{t+1}}{w_t - s_t} = (\beta R_{t+1})^{1/\theta}$$

- savings:

$$s_t = \frac{w_t}{1 + \beta^{-1/\theta} R_{t+1}^{1-1/\theta}}$$

note: $1 + \beta^{-1/\theta} R_{t+1}^{1-1/\theta} > 1 \rightarrow$ savings always less than income

Saving Function: Income and Substitution Effects

- impact of factor prices on savings $s = \frac{w}{1+\beta^{-\frac{1}{\theta}} R^{\frac{\theta-1}{\theta}}}$:

$$\frac{\partial s}{\partial w} > 0$$

$$\frac{\partial s}{\partial R} > 0 \quad \text{if} \quad \theta < 1 \quad \text{and} \quad \frac{\partial s}{\partial R} < 0 \quad \text{if} \quad \theta > 1$$

- two opposite effects of R on savings:
  1. substitution effect: higher return to saving tend to increase s
  2. income effect: higher (capital) income tend to increase both c_{1t} and c_{2t+1} ($\downarrow s$)

- if θ is low, agents substitute consumption across dates easily $\longrightarrow$ substitution effect dominates

- note: if $\theta = 1$ (log utility) the two effects cancel out
  $\longrightarrow$ savings independent of the interest rate (as in Solow)

Equilibrium Dynamics: Simple Case

- assume:

$$\begin{aligned} f(k_t) &= k_t^\alpha \longrightarrow w_t = k_t^\alpha - k_t \alpha k_t^{\alpha-1} = (1-\alpha)\, k_t^\alpha \\ \theta &= 1 \longrightarrow s_t = \frac{\beta w_t}{1+\beta} \end{aligned}$$

- fundamental equation:

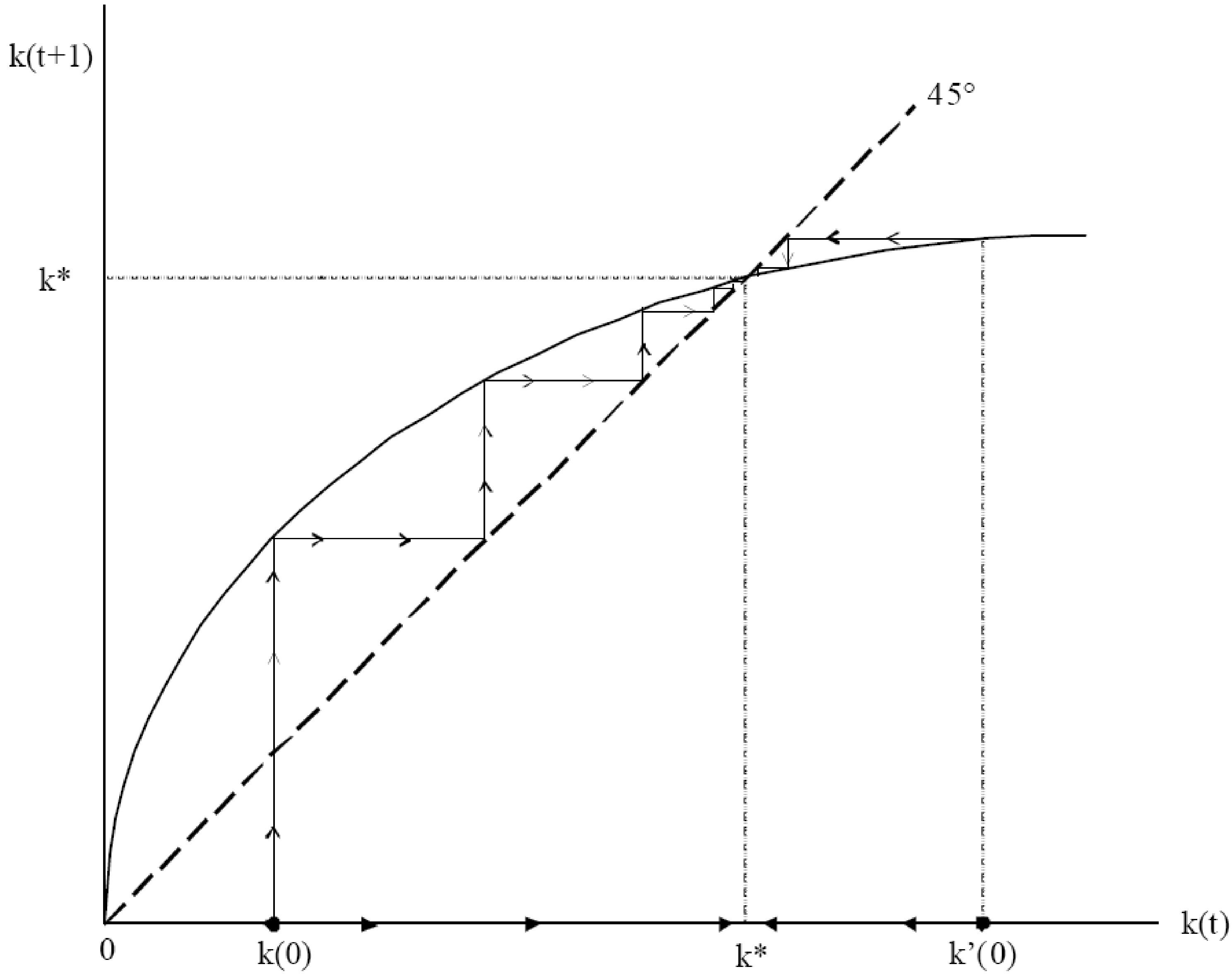
$$k_{t+1} = \frac{s(w_t)}{1+n} = \frac{\beta(1-\alpha)\, k_t^\alpha}{(1+\beta)(1+n)}$$

- converges to the steady-state $(k_{t+1} = k_t = k)$:

$$k = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

as in the Solow model

Equilibrium Dynamics: Graph



The Golden Rule

- from the resource constraint:

$$\begin{aligned} F(K_t, L_t) &= K_{t+1} + L_t c_{1t} + L_{t-1} c_{2t} \rightarrow \\ \frac{F(K_t, L_t)}{L_t} &= f(k_t) = (1+n)\, k_{t+1} + c_{1t} + \frac{c_{2t}}{1+n} \end{aligned}$$

in steady-state:

$$f(k^*) = (1+n)\, k^* + c^*$$

where c^* denotes *total* consumption per worker in steady-state

- c^* is maximized for:

$$f'\left(k_{gold}\right) = 1 + n$$

or, using $f'(k_t) = 1 + r_t$:

$$r^{gold} = n$$

$\rightarrow$ the golden rule of the Solow model

Pareto Optimality and Overaccumulation

- in the OLG model, steady-state capital may be above the golden rule level:

$$r^* < n$$

- for example, in the Cobb-Douglas case we have:

$$k_{gold} = \left(\frac{\alpha}{1+n}\right)^{\frac{1}{1-\alpha}} \quad \text{thus} \quad k^* > k_{gold} \text{ if}$$

$$\left[\frac{\beta\,(1-\alpha)}{(1+\beta)\,(1+n)}\right]^{\frac{1}{1-\alpha}} > \left(\frac{\alpha}{1+n}\right)^{\frac{1}{1-\alpha}} \Leftrightarrow \frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$$

- if $r^* < n$ the OLG equilibrium is *inefficient:*
  if we reduce k_t marginally $\longrightarrow$ we can increase c_t (total consumption) at all dates
  we can always redistribute the higher c_t among generations to make all of them better off

Why May Agents Overaccumulate?

- saving is the only way to consume when old

- suppose β is high and α (capital share) is low:
  strong desire to consume when old, but low return $\longrightarrow$ very high saving

- agent willing to accept a lower lifetime consumption to smooth it

- however, given the aggregate resource constraint, the economy could do better by:
  1. reducing k to k_{gold} $\longrightarrow$ (allows to consume more immediately)
  2. use transfers of c^* (from the young to old) to sustain consumption of the old (optimal lifetime smoothing)

- note: the argument does not apply if $k^* < k_{gold}$ because it would require *less* consumption for the first generation

Fully-Funded Social Security

- government takes d_t from the young, invests it in capital, and pays back $R_{t+1}d_t$ when old

- individual maximization:

$$\begin{aligned} \max U_t &= u(c_{1t}) + \beta u(c_{2t+1}) \\ st &: c_{1t} + s_t + d_t \leq w_t \\ &: c_{2t+1} \leq (s_t + d_t) R_{t+1} \end{aligned}$$

- solution:

$$\begin{aligned} u'(w_t - s_t - d_t) &= \beta u'((s_t + d_t) R_{t+1}) \\ k_{t+1} &= \frac{s_t + d_t}{1 + n} \end{aligned}$$

same equations as before: as long as $s_t > 0 \rightarrow d_t$ has no effect on k_{t+1}!

Unfunded Social Security (pay-as-you-go)

- government collects d_t from the young at t and distributes to the current old with per capita transfer:

$$b_t = \frac{d_t L_{t+1}}{L_t} = (1+n)d_t$$

- individual maximization:

$$\begin{aligned} \max U_t &= u(c_{1t}) + \beta u(c_{2t+1}) \\ st &: c_{1t} + s_t + d_t \leq w_t \\ &: c_{2t+1} \leq s_t R_{t+1} + (1+n)d_{t+1} \end{aligned}$$

note: the "return" from social security is $(1+n)$ rather than $(1+r_{t+1})$

Unfundded Social Security: Solution

- substituting c_{1t} and c_{2t+1} into the Euler equation:

$$u'(w_t - s_t - d_t) = \beta u'(s_t R_{t+1} + (1+n)d_t)$$

law of motion of k_{t+1}

$$k_{t+1} = \frac{s_t}{1+n}$$

note: d_t is a transfer $\longrightarrow$ does not go into capital accumulation

- effect of d_t on savings for given prices (w_t, R_{t+1})
from implicit differentiation $-u_1''\partial s - u_1''\partial d = \beta u_2'' R_{t+1}\partial s + \beta u_2''(1+n)\partial d$:

$$\frac{\partial s}{\partial d} = -\frac{u_1'' + \beta u_2''(1+n)}{u_1'' + \beta u_2'' R_{t+1}} < 0$$

- d_t lowers savings $\longrightarrow$ lower k^* $\longrightarrow$ may correct overaccumulation

Conclusions

- OLG may be more realistic than infinite horizon models

- dynamics: similar to Solow

- main new result:
  - the CE may be dynamically inefficient
  - if so, an unfunded social security system may be welfare improving

- yet, overaccumulation is probably not so relevant in reality

- new insights on the sources of income differences: no, but OLG and finite lives are useful to study other problems, such as investment in human capital

Advanced Macro - Lecture 7

Human Capital and Economic Growth

Ref: Acemoglu ch. 10

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Human Capital and Economic Growth

- *human capital* refers to all attributes of workers that:
  1. can be accumulated through investment
  2. increase their productivity

- the theory of human capital is at the core of labor economics

- human capital accumulation may also be a key factor in explaining long-run growth and cross-country income differences

- we will discuss three models:
  1. Mincer/Ben-Porath model of schooling decisions
  2. Neoclassical model with human capital accumulation
  3. Nelson-Phelps model where human capital facilitates technology adoption

Determinants of Schooling Investment

- simple version of the Mincer/Ben-Porath model
- highlights the determinants of the schooling decision
- partial equilibrium model: agents take prices as given
- continuous time
- perfect capital markets
- utility:

$$\max \int_0^\infty e^{-(\rho+\nu)} u\left(c_t\right) dt$$

$\rho =$ discount factor
$\nu =$ instantaneous death probability

Budget Constraint

- new problem: agents may die with positive assets

- simple solution:
  introduce an insurance market that pays a flow of $z(a_t)$ for the right to seize asset holding a_t upon death
  zero profit for the insurance company requires:

$$\nu a_t - z(a_t) = 0 \rightarrow z(a_t) = \nu a_t$$

- then, the flow budget constraint is:

$$\begin{aligned} \dot{a}_t &= (r + \nu)\, a_t + w_t h_t - c_t \\ a_0 &= 0 \end{aligned}$$

  where $h_t =$ human capital and $w_t =$ wage per unit of h

Effect of Schooling

- schooling decision: choose the length S of the time interval spent in school

- human capital after schooling period of S:

$$h = \eta(S)$$

  with $\eta'(S) > 0$, $\eta''(S) < 0$

- human capital may keep growing after $t = S$ at an exogenous rate $g_h \geq 0$ (experience):

$$\dot{h}_t = g_h h_t$$

- wages per unit of h grow exogenously at rate $g_w \geq 0$:

$$\dot{w}_t = g_w w_t$$

  thus, total wage at t given S:

$$W_{t,S} = w_t h_t = w_0 e^{g_w t} \eta(S) e^{g_h (t-S)}$$

Schooling and Consumption Decisions: Separation Theorem

- under complete markets $\longrightarrow$ optimal schooling decision maximizes the present discounted value of income

- thus, schooling decision separated from consumption decision

- intuitive: with complete markets, the optimal schooling decision maximizes the agent's budget set while consumption smoothing is achieved through borrowing and lending

- thus:

$$\begin{aligned}\max_S \int_S^\infty e^{-(r+\nu)t} w_t h_t dt &= w_0 \eta(S) e^{-g_h S} \int_S^\infty e^{-(r+\nu-g_w-g_h)t} dt = \\ &= w_0 \eta(S) e^{-g_h S} \left[\frac{-e^{-(r+\nu-g_w-g_h)t}}{r+\nu-g_w-g_h} \right]_S^\infty = \\ &= \frac{\eta(S) w_0 e^{-(r+\nu-g_w)S}}{r+\nu-g_h-g_w}\end{aligned}$$

Determinants of Schooling

- take logs and maximize:

$$\max_S \{\ln \eta(S) + \ln w_0 - (r + \nu - g_w)\, S - \ln (r + \nu - g_h - g_w)\}$$

$$\rightarrow \quad FOC : \frac{\eta'(S^*)}{\eta(S^*)} = r + \nu - g_w$$

- high r and ν reduces the value of schooling (schooling pays out in the future)

- schooling independent of w (both opportunity costs and benefit of schooling)

- wage *growth* (technical progress) increases schooling
  $g_w > 0$ lowers the opportunity cost of schooling relative to its benefit

- suggests a complementarity between education and growth (adding a feedback education $\rightarrow$ growth)

Cross-Sectoral Prediction: Mincer equation

- integrate $\frac{\eta'(S^*)}{\eta(S^*)} = r + \nu - g_w$ to solve for S^*, substitute into $W_{t,S} = w_0 e^{g_w t} \eta(S) e^{g_h(t-S)}$ and take logs:

$$\ln W_{t,S^*} = \bar{c} + (r + \nu - g_w)\, S^* + g_w t + g_h(t - S)$$

- in a cross-section of workers, the time trend $g_w t$ goes into the constant

- Mincer equation:
  log wage earnings are proportional to schooling and experience

$$\ln W_j = const + \gamma_s \text{ schooling} + \gamma_s \text{ experience}$$

- these equations are frequently estimated by labor economists

Neoclassical Growth with Physical and Human Capital

- extend the neoclassical growth model by introducing human capital accumulation

- simplification: human capital can be accumulated precisely as physical capital
  $\longrightarrow$ human capital more similar to "knowledge" rather than schooling

- continuous time, infinite horizon

- ignore technological progress and population growth

Preferences and Technology

- neoclassical production function: $Y_t = F(K_t, H_t, L)$
  where H_t is total human capital $\rightarrow H$ and L are different factors
  $$y_t = \frac{Y_t}{L} = f(k_t, h_t)$$
  where $h_t = H_t / L$

- from now, remove the time index to simplify notation

- law of motion of k and h :
  $$\begin{aligned} \dot{h} &= i_h - \delta_h h \\ \dot{k} &= i_k - \delta_k k \end{aligned}$$
  i_h and $i_k =$ investment in human and physical capital, δ depreciation

- resource constraint: $c + i_h + i_k \leq f(k_t, h_t)$

Optimal Growth (= CE)

- current value Hamiltonian:

$$\hat{H} = u\left(f - i_h - i_k\right) + \mu_h(i_h - \delta_h h) + \mu_k(i_k - \delta_k k)$$

  note: 2 state variables (h, k), 2 costates (μ_h and μ_k), 2 controls (i_h, i_k)

- FOCs:

$$\begin{aligned}
\hat{H}_{i_h} &= -u'(c) + \mu_h = 0 \\
\hat{H}_{i_k} &= -u'(c) + \mu_k = 0 \\
\hat{H}_h &= u'(c) f_h - \mu_h \delta_h = \rho\mu_h - \dot{\mu}_h \\
\hat{H}_k &= u'(c) f_k - \mu_k \delta_k = \rho\mu_k - \dot{\mu}_k \\
\lim_{t\to\infty} e^{-\rho t}\mu_{h,t}h_t &= 0 \\
\lim_{t\to\infty} e^{-\rho t}\mu_{k,t}k_t &= 0
\end{aligned}$$

Analysis

- from $\hat{H}_{i_h} = \hat{H}_{i_k} = 0$:

$$\mu_h = \mu_k = u'(c)$$

the other two FOCs imply:

$$f_h - \delta_h = f_k - \delta_k$$

implicitly pins down $h = \xi(k)$

- Euler equation: $$\frac{\dot{c}}{c} = \frac{f_k(k,\xi(k)) - \delta_k - \rho}{\theta}$$

where $\theta = -\frac{u''(c)c}{u'(c)}$

- transversality requires: $\lim_{t\to\infty} k_t e^{-f_k(k,\xi(k))} = 0$

note: k and h always in balance, dynamics similar to the model without h

Example

- make functional form assumptions:

$$\begin{aligned} Y &= AK^{\alpha}H^{\beta}L^{1-\alpha-\beta} \rightarrow y = Ak^{\alpha}h^{\beta} \\ u(c) &= \frac{c^{1-\theta}}{1-\theta} \rightarrow u'(c) = c^{-\theta} \\ \delta_h &= \delta_k = \delta \end{aligned}$$

to get:

$$\begin{aligned} f_h &= f_k \rightarrow h = k\frac{\beta}{\alpha} \\ \frac{\dot{c}}{c} &= \frac{1}{\theta}\left(\alpha^{1-\beta}\beta^{\beta}Ak^{\alpha+\beta-1} - \delta - \rho\right) \end{aligned}$$

- note:

1. $\alpha + \beta < 1 \rightarrow$ converges to a unique steady-state with $k = \left(\frac{\alpha^{1-\beta}\beta^{\beta}A}{\delta+\rho}\right)^{\frac{1}{1-\alpha-\beta}}$
2. endogenous growth if $\alpha + \beta = 1$ and $\alpha^{\alpha}(1-\alpha)^{1-\alpha}A > \delta + \rho$
3. transversality satisfied when $r > \frac{\dot{c}}{c}$ or $\rho > (1-\theta)\left[\alpha^{\alpha}(1-\alpha)^{1-\alpha}A - \delta\right]$

Neoclassical Model with K and H: Comments

- the fact that k and h are always in balance is due to the absence of non-negativity constraints on investment

- if investment cannot be negative, out of steady-state there will be periods in which only h or k grows, until the optimal balance is restored

- we can obtain endogenous growth if production is linear in the combination of the two accumulable factors $(\alpha + \beta = 1) \rightarrow$ "AK models"

- yet, in reality the share of accumulable factors is much below one

- are we missing something? human capital may create *externalities* that can sustain endogenous growth even when the share of accumulable factors is less than one

- yet the evidence suggests human capital externalities to be small

Alternative Perspective: Nelson and Phelps

- human capital not only increase productivity in existing tasks, but enable workers to learn how to use new technologies

- Nelson and Phelps (1966):
  - simple continuous time model to capture this effect
  - mechanical, but influential idea

- production: $Y_t = A_t L$
  where A_t = productivity = technology

- there is a world technology frontier A_t^* that grows exogenously

$$\dot{A}_t^* = g^* A_t^*$$

- human capital h does not enter into Y directly, but affects the rate at which a country can adopt frontier technologies

Nelson and Phelps: Human Capital and Technology Gaps

- technology in use A_t evolves according to

$$\dot{A}_t = gA_t + \phi(h)A_t^*$$

where $g < g^*$ is exogenous, h is (constant) human capital and

$$\phi'(h) > 0,\ \phi(0) = 0 \text{ and } \max \phi(h) = g^* - g$$

Rearrange:

$$\frac{\dot{A}_t}{A_t} = g + \phi(h)\frac{A_t^*}{A_t}$$

as $t \to \infty$, $\dot{A}_t/A_t = g^*$ or else the second term will grow either to zero or infinity

- using $\dot{A}_t/A_t = g^*$:

$$g^* = g + \phi(h)\frac{A_t^*}{A_t} \to A_t = \frac{\phi(h)}{g^* - g}A_t^*$$

the productivity/income gap depends on human capital

Conclusions

- physical and human capital accumulation respond to current and future incentives

- physical and human capital accumulation seems to be complementary

- human capital may create externalities or speed up technology adoption

- the next step is to endogenize the third input in the neoclassical production function: technology/productivity

Advanced Macro - Lecture 8

Endogenous Technical Change

Ref:
Gancia and Zilibotti (2005) 1-2
Acemoglu ch. 13
Barro and Sala-i-Martin ch.6

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Endogenous Technical Change: Introduction

- so far, technical progress was taken as exogenous
- yet, R&D and technology adoption are purposeful activities
- simplest models of endogenous technical change: horizontal innovation
- innovation = introduction of new goods that do not displace existing ones
  underlying assumption: availability of more goods raises productivity/utility
- how? two approaches:
  - "returns form specialization" in production: $Y_t = L_{y,t}^{1-\alpha} \int_0^{A_t} x_{j,t}^{\alpha} dj,$
  - "love of variety" in consumption: $U_t = \left(\int_0^{A_t} x_{j,t}^{\alpha} dj \right)^{1/\alpha}$

  where A_t is the measure of available goods

Modelling Innovation: Key Aspects

- innovation = idea = design of a new product

- ideas have two key features:
  1. non-rival: an existing idea (once discovered) can be replicated at no cost
  2. partly excludable: patents may prevent others from exploiting economically ideas

- to capture these features, we assume that innovation:
  1. is a fixed cost
  2. grants monopoly rights to the innovator on the use of his discovery

- key role of monopoly profits:
  - the prospect of profits ex-posts motivate the ex-ante investment in R&D

- new difficulty: need to abandon perfect competition
  take the smallest departure: monopolistic competition with free-entry

Benchmark Model of Horizontal Innovation

- simplified version of Romer (1990) with no physical capital

- horizontal innovation = introduction of new product variety that does not displace existing varieties

- continuous time, infinite horizon, no population growth

- two versions:

   1. knowledge spillover: cost of innovation in units of labor

   2. lab-equipment: cost of innovation in units of final output

Structure of the Model (t omitted)

- households: supply saving $\longrightarrow$ Euler equation : $r = \rho + \theta g$

- final good sector (competitive):
  demand intermediates to assemble final goods

$$Y = L_y^{1-\alpha} \int_0^A x_i^\alpha di \quad \longrightarrow \quad x_i = x(p_i)$$

- intermediate good sector (monopolistic):
  each firm produces and sells one variety of intermediates

$$\max_{p_i} \pi_i \quad \text{s.t.} \quad x(p_i)$$

- R&D sector: discovers of new intermediates at cost μ
  value of a new intermediate = PDV of future profits = $V(\pi, r)$
  up-front cost + future payoff $\longrightarrow$ need to borrow
  free entry in R&D: $V(\pi, r) = \mu$ (value = cost of R&D)

Households

- L infinitely lived agents, inelastic labor supply

- consumption path is set to maximize:

$$\begin{aligned} \max_c U &= \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \\ st &: \dot{b}_t = r_t b_t + w_t - c_t \\ & \quad \text{No-Ponzi condition} \end{aligned}$$

- from the FOCs we obtain the Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

and the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} \mu_t a_t = \lim_{t\to\infty} e^{-\bar{r}_t t} a_t = 0$$

Final Good Sector (competitive) - FGS

- employs labor and intermediates as inputs. Production function:

$$Y_t = L_{y,t}^{1-\alpha} \int_0^{A_t} x_{j,t}^{\alpha} \, dj, \quad \alpha \in (0,1)$$

  $x_{j,t}$ = quantity of the intermediate j
  A_t = measure of intermediates available at t
  $L_{y,t}$ = labor employed in FGS
  Note: different inputs are imperfect substitutes and enter symmetrically

- profit maximization of the representative firm:

$$\max \left\{ L_{y,t}^{1-\alpha} \int_0^{A_t} {x_{j,t}}^{\alpha} dj - w_t L_{y,t} - \int_0^{A_t} p_{j,t} x_{j,t} dj \right\}$$

  p_j = price of variety j

$$\begin{aligned} \frac{\partial \{\cdot\}}{\partial x_{j,t}} &= 0 \to p_{j,t} = \alpha L_{y,t}^{1-\alpha} \left(x_{j,t} \right)^{\alpha-1} \qquad \forall j \\ \frac{\partial \{\cdot\}}{\partial L_{y,t}} &= 0 \to w_{y,t} = (1-\alpha) L_{y,t}^{-\alpha} \int_0^{A_t} {x_{j,t}}^{\alpha} dj \end{aligned}$$

Intermediate Good Sector (monopolistic) - IGS

- each variety is produced by a single firm

- technology: one unit of intermediate good requires one unit of final good

- profit maximization:

$$\max_{p_{j,t}} \left\{ p_{j,t} x_{j,t} - x_{j,t} \right\}$$

$$s.t. \quad : \quad x_{j,t} = \alpha^{\frac{1}{1-\alpha}} L_{y,t} p_{j,t}^{\frac{1}{\alpha-1}}$$

note: demand for $x_{j,t}$ has a constant price elasticity of $\epsilon = \frac{1}{\alpha-1}$

- monopoly pricing:

$$p_{j,t} \left(1 - \frac{1}{|\epsilon|} \right) = \text{MC} \rightarrow p_{j,t} = p = \frac{1}{\alpha}$$

all monopolists charge the same price

Profits and Wages

- substituting $p = 1/\alpha$ into demand:

$$x_{j,t} = x_t = \alpha^{\frac{2}{1-\alpha}} L_{y,t}$$

  and x_t into wages:

$$w_t = (1 - \alpha)\alpha^{\frac{2\alpha}{1-\alpha}} A_t$$

  wages are linear in A_t

- profits:

$$\pi_{j,t} = \pi_t = (p - 1)x_t = \frac{1 - \alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} L_{y,t}$$

  do not depend on A_t

Cost of Innovation

- designing a new variety requires a sunk cost of $1/\delta A_t$ units of labor
  $\longrightarrow$ cost of innovation:
  $$\mu \equiv \frac{w_t}{\delta A_t}$$
  knowledge spillover:
  productivity in R&D increases with the stock of "knowledge", A_t
  researchers benefit from past discoveries ("standing on the shoulders of giants")

- law of motion of A_t:
  $$\dot{A}_t = \delta A_t L_{RD,t}$$
  $L_{RD,t}$ = labor employed in R&D
  note: the rate of innovation is a linear function of employment in R&D

Value of Innovation

- value of innovation = PDV of future profits:

$$V_t = \int_t^{\infty} e^{-\bar{r}_{s-t}(s-t)} \pi_s ds$$

  or, in the Hamilton-Jacobi-Bellman form:

$$r_t = \frac{\pi_t + \dot{V}_t}{V_t}$$

  $\longrightarrow$ return from R&D = market rate r (arbitrage condition)

- free entry: $V_t \leq \mu$

Balanced Growth Equilibrium

- guess-and-verify a balanced growth (BG) equilibrium where c_t, Y_t, w_t and A_t grow at the constant rate, g
  $\longrightarrow$ along this BG x_t, π_t, r_t and $L_{RD,t}$ are constant
  $\longrightarrow$ the value of innovation has to be constant too: $\dot{V}_t = 0 \longrightarrow V = \frac{\pi}{r}$

- if $g > 0$, free-entry in R&D requires:

$$\text{value of innovation } = \frac{\pi}{r} = \frac{w_t}{\delta A_t} = \text{ cost of innovation}$$

  note: the right hand side is constant in BG due to the knowledge externality (w_t and A_t grow at the same rate)

- labor market clearing:

$$\begin{aligned} L &= L_y + L_{RD} \\ \dot{A}_t &= \delta A_t L_{RD} \rightarrow L_{RD} = g/\delta \end{aligned}$$

Balanced Growth: Solution

- the growth rate can be found solving the system:

$$\begin{aligned} \text{free entry} &: \frac{\pi}{r} = \frac{w_t}{\delta A_t} \\ \text{Euler equation} &: g = \frac{r-\rho}{\theta} \\ \text{full employment} &: L = L_y + \frac{g}{\delta} \end{aligned}$$

- using $\pi = \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}}L_{y,t}$ and $w_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}}A_t$:

$$g = \frac{\delta\alpha L - \rho}{\alpha + \theta}$$

  $g > 0$ if $\alpha\delta L > \rho$

- g is increasing in R&D productivity δ, the size of the labor force L (scale effect) and decreasing in θ and ρ

Balanced Growth and Transversality

- final output:

$$Y_t = L_y^{1-\alpha} \int_0^{A_t} x_j^{\alpha} \, dj = A_t L_y^{1-\alpha} x^{\alpha} dj$$

  linear in $A_t \rightarrow \dot{Y}_t / Y_t = \dot{A}_t / A_t$ (x is constant)

- aggregate resource constraint:

$$C_t + A_t x = Y_t \rightarrow C_t = A_t \left(L_y^{1-\alpha} x^{\alpha} - x \right)$$

  this verifies that C_t and A_t also grow at the same rate

- transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\bar{r}_t t} a_t = \lim_{t \rightarrow \infty} e^{-\bar{r}_t t} A_t V_t = 0$$

  requiring $r > g \rightarrow \rho > g\,(1-\theta) \rightarrow \rho > \frac{(\delta \alpha L - \rho)(1-\theta)}{\alpha+\theta}$

Optimal Growth (Social Planner Solution)

- social planner problem:

$$\begin{aligned} \max U &= \int_0^\infty e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} dt \\ s.t. &: Y_t = C_t + A_t x_t \\ &: \dot{A}_t = \delta \left(L - L_{y,t} \right) A_t \end{aligned}$$

Hamiltonian: $H = \frac{C_t^{1-\theta}-1}{1-\theta} + \mu_t \left(A_t x_t^\alpha L_{y,t}^{1-\alpha} - A_t x_t - C_t \right) + \lambda_t \left[\delta \left(L - L_{y,t} \right) A_t \right]$

- the CE is not optimal (you are asked to verify this in PS3)

- why? two distortions:
  1. monopoly pricing $\rightarrow$ underproduction of each x_j
  2. R&D produce externalities (knowledge spillover) that innovators ignore

- CE is inefficient and growth sub-optimally low

“Lab-Equipment” version (Romer & Rivera Batiz, 1991)

- alternative formulation of the R&D cost:
  research uses final output instead of labor as a productive input
  $\longrightarrow$ designing a new variety requires μ units of Y:
  all workers employed in the FGS

- key equilibrium conditions:

$$\begin{aligned} \text{Free entry} &: \quad \frac{\pi}{r} = \mu \\ \text{Euler equation} &: \quad g = \frac{r - \rho}{\theta} \\ \text{Full employment} &: \quad L_y = L \end{aligned}$$

“Lab-Equipment”: Solution

- using π and w_t yields:

$$g = \frac{1}{\theta}\left[(1-\alpha)\,\alpha^{\frac{1+\alpha}{1-\alpha}}\frac{L}{\mu} - \rho\right]$$

  determinants of grows similar to previous version

- note: $\dot{A}_t = Y_{RD}/\mu$, where Y_{RD} = units of final output devoted to R&D (hence, $C = Y - Ax - Y_{RD}$)

- sustained growth is attained by allocating a constant share of production to R&D, no spillover

- slightly more tractable model

Conclusions

- key elements: ideas are non-rival + monopolistic competition
- similar to AK models, but very different economic foundation
- technical progress is determined by incentives:
  market structure, competition policy, IPR protection
- equilibrium typically not Pareto optimal
- limit: Schumpeterian aspects of innovation like creative destruction and quality differentiation missing

Advanced Macro - Lecture 9

Directed Technical Change and Applications

Ref:
Gancia and Zilibotti (2008)
Gancia and Zilibotti (2005) section 4
Acemoglu ch. 15

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Directed Technical Change

- so far, we have focused on technology as Hicks-neutral TFP

- yet, technological change is often not neutral: it may benefit some factor of production/agents more than others
  $\longrightarrow$ important distributional effects

- examples:
  1. in the past 60 years, new technologies has been skill-biesed (eg, computers)
  2. in the 19th century, new technologies were unskilled-biased (eg, chain production)

- Directed Technical Change (DTC) endogenizes the bias/direction of new technologies

Directed Technical Change and Income Distribution

- by shifting the emphasis from the *level* of technology to its *shape*, DTC is useful to understand the distributional effects of technology:

  skill-premia within coutries

  income differenecs between countries

- indeed, technology is a major determinants of skill-preima

- and productivity (technology?) is also the main source of income differences

- sustained technological change created prosperity in some countries but also enormous inequality in the wealth of nation

Directed Technical Change: Applications

- Acemoglu (1998):
  introduces DTC to explain why new technologies have been skill-complement in the recent past

- Acemoglu and Zilibotti (2001):
  uses DTC to shows that technologies developed in advanced countries are inappropriate for the skill-endowment of poor countries $\longrightarrow$ low TFP

- Acemoglu (2002)/Epifani and Gancia (2008):
  argues that trade may spur skill-biased technical change

- we follow Gancia and Zilibotti (2005, 2008):
  tractable model of DTC with many applications

Directed Technical Change: Basic Idea

- extension of Romer (1990), lab-equipment version
- 2 factors of production skilled (H) and unskilled (L) workers
- skilled and unskilled workers produce different goods and use different "technologies" (=intermediates)
- the production function in *each* sector (H and L) is like in models of horizontal innovation
- the development of H- or L-complement innovations is endogenous and depends on relative profitability

Benchmark Model (time index omitted)

- 2 factors of production, H (skilled labor) and L (unskilled labor), no capital

- 2 sets of countries: North (innovates) and South (adopts)

- for now: focus on North only

- aggregate production function:

$$Y = \left(Y_L^{\frac{\epsilon-1}{\epsilon}} + Y_H^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1$$

  $Y_L =$ goods produced with L
  $Y_H =$ goods produced with H
  $\epsilon =$ elasticity of substitution between Y_L and Y_H

Demand for Y_L and Y_H

- recall: $$Y = \left(Y_L^{\frac{\epsilon-1}{\epsilon}} + Y_H^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$$

- profit maximization gives constant elasticity demand functions:

$$\max_{Y_H, Y_L} \quad \{Y - P_H Y_H - P_L Y_L\}$$

$$FOCs \quad : \quad Y^{\frac{1}{\epsilon}} Y_H^{\frac{-1}{\epsilon}} = P_H \quad \text{and} \quad Y^{\frac{1}{\epsilon}} Y_L^{\frac{-1}{\epsilon}} = P_L$$

$$\rightarrow \quad \frac{Y_L}{Y_H} = \left(\frac{P_H}{P_L}\right)^{\epsilon}$$

where P_L and P_H are the prices of Y_L and Y_H, respectively

- note:
  1. demand is a negative function of price
  2. price elasticity of demand $= \epsilon$

Sectorial Production: Y_L and Y_H (Competitive)

- sectorial production functions:

$$
\begin{aligned}
Y_L &= E_L \left[\int_0^{A_L} y_L(i)^{\frac{\sigma-1}{\sigma}} \, di \right]^{\frac{\sigma}{\sigma-1}} \\
Y_H &= E_H \left[\int_0^{A_H} y_H(i)^{\frac{\sigma-1}{\sigma}} \, di \right]^{\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

  note:
  1. Y_L and Y_H are produced with *sector-specific* intermediates
  2. A_L and A_H capture the state of technology in the two sectors

- aggregate externality:

$$
E_L \equiv (A_L)^{\frac{\sigma-2}{\sigma-1}} \quad \text{and} \quad E_H \equiv (A_H)^{\frac{\sigma-2}{\sigma-1}}
$$

  to guarantee balanced growth

- Y_L and Y_H are produced by competitive firms

Demand for Intermediates

- focus on production of Y_H (Y_L equivalent, after relabeling)

- define $p_H(i)$ as the price of intermediate $y_H(i)$

- profit maximization (taking P_H and $p_H(i)$ as given)

$$\max_{y_H(i)} \left\{ P_H Y_H - \int_0^{A_H} p_H(i)\, y_H(i)\, di \right\}$$

  yields relative demand for intermediates:

$$\frac{y_H(i)}{y_H(j)} = \left[\frac{p_H(j)}{p_H(i)}\right]^{\sigma}$$

- note: price elasticity of demand $\frac{-\partial \ln y_H(i)}{\partial p_H(i)} p_H(i)$ is σ

Intermediate Good Sector: $y_L(i)$ **and** $y_H(i)$ **(Monopolistic)**

- each intermediate is produced by a monopolist with linear technology:

$$y_L(i) = l(i) \quad \text{and} \quad y_H(i) = Zh(i)$$

- markup pricing:

$$p_H = \left(1 - \frac{1}{\sigma}\right)^{-1} \frac{w_H}{Z} \quad \text{and} \quad p_L = \left(1 - \frac{1}{\sigma}\right)^{-1} w_L$$

- imposing symmetry and labor market clearing:

$$l(i) = L/A_L \text{ and } h(i) = H/A_H$$

into production:

$$\begin{aligned} Y_L &= (A_L)^{\frac{\sigma-2}{\sigma-1}} \left[\int_0^{A_L} y_L(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} = A_L L \\ Y_H &= A_H Z H \end{aligned}$$

Wages and the Skill Premium

- markup pricing implies:

$$w_H h(i) = \left(\frac{\sigma - 1}{\sigma}\right) p_H h(i) \quad \text{and} \quad \pi_H(i) = \frac{p_H h(i)}{\sigma}$$

  a fraction $1/\sigma$ of revenue goes into profits the rest into wages

- using $p_L L = P_L Y_L$, $p_H Z H = P_H Y_H$ and relative demand P_H/P_L:

$$\frac{w_H}{w_L} = \frac{p_H}{p_L} = \frac{P_H Y_H / ZH}{P_L Y_L / L} = \left(\frac{Z A_H}{A_L}\right)^{1-\frac{1}{\epsilon}} \left(\frac{L}{H}\right)^{\frac{1}{\epsilon}}$$

  note:

  1. ϵ is the elasticity of substitution between H and $L : \epsilon = -\frac{d\ln(H/L)}{d\ln(w_H/w_L)}$
  2. the skill premium, w_H/w_L, is decreasing in the relative supply of skilled labor (H/L) and increasing in the skill-bias (A_H/A_L) when $\epsilon > 1$

Relative Profits

- profits = fraction $1/\sigma$ of sectorial revenue
  by symmetry, $p_H h\left(i\right) = P_H Y_H / A_H$ thus:

$$\frac{\pi_H}{\pi_L} = \frac{P_H Y_H / A_H}{P_L Y_L / A_L} = \frac{P_H Z H}{P_L L} = \left(\frac{A_H}{A_L}\right)^{-\frac{1}{\epsilon}} \left(\frac{ZH}{L}\right)^{1-\frac{1}{\epsilon}}$$

  where I used relative demand to substitute P_H/P_L

- as in Romer (1990) profits are the reward of innovation
  $\rightarrow \frac{\pi_H}{\pi_L}$ is also the relative profitability of R&D directed to the two sectors

- components of profitability:
  1. "price effect": higher incentive to invent technologies producing more expensive goods
  2. "market size" effect: the incentive to develop a new technology is proportional to the number of workers that will be using it

Directed Technical Change

- innovation = introduction of new intermediates

- value of innovation = V = PDV of future profit stream
  in balanced growth r, π_L and π_H will be constant:

$$V_L = \frac{\pi_L}{r} \quad \text{and} \quad V_H = \frac{\pi_H}{r}$$

- cost of one innovation = μ units of Y

- free entry:

$$V_L \leq \mu \quad \text{and} \quad V_H \leq \mu$$

  with equality if innovation is positive

Endogenous Skill-Bias (Acemoglu 1998)

- focus on the balanced growth path (BGP)

- BG requires $V_L = V_H = \mu \rightarrow \pi_H = \pi_L$:

$$\frac{\pi_H}{\pi_L} = \left(\frac{A_H}{A_L}\right)^{-\frac{1}{\epsilon}} \left(\frac{ZH}{L}\right)^{1-\frac{1}{\epsilon}} = 1$$

$$\rightarrow \frac{A_H}{A_L} = \left(\frac{ZH}{L}\right)^{\epsilon-1}$$

  note: when $\epsilon > 1$ innovation is biased in favor of the abundant factor

- what happens outside the BGP?
  $\rightarrow$ one type of innovation only until we converge to $\frac{A_H}{A_L} = \left(\frac{ZH}{L}\right)^{\epsilon-1}$ (ie. the BGP is stable)

Endogenous Skill-Premium (BGP)

- using $\frac{A_H}{A_L}$ into the skill-premium

$$\frac{w_H}{w_L} = \left(\frac{ZA_H}{A_L}\right)^{1-\frac{1}{\epsilon}} \left(\frac{L}{H}\right)^{\frac{1}{\epsilon}} = \left[\frac{H}{L}\right]^{\epsilon-2}$$

  note: relationship between relative wages and relative labor supply can either be positive or negative
  1. given $\frac{A_H}{A_L}$, a large supply of one factor depresses its price
  2. yet, a large supply of one factor induces a technology bias in its favor, thereby raising its productivity

- the strenght of the two effects depend on ϵ :
  high substitutability $\longrightarrow$ weak price effect $\longrightarrow$ positive relationship more likely
  if $\epsilon > 2 \longrightarrow$ an increase in supply leads to an increase in price!
  this may help explain the recent increase in the skill premium in US/UK

Solving for the Growth Rate g

- as in Romer (1990), the growth rate is found using $r = \pi_H/\mu$ (or $r = \pi_L/\mu$) plus Euler equation: With log utility:

$$g = r - \rho = \frac{\pi_H}{\mu} - \rho = \frac{P_H Z H}{\mu} - \rho$$

- to solve for P_H, combine $Y = P_H Y_H + P_L Y_L$ and relative demand $\frac{Y_H}{Y_L}$:

$$P_L^{1-\epsilon} + P_H^{1-\epsilon} = 1 \rightarrow P_H = \left[\left(\frac{P_H}{P_L}\right)^{\epsilon-1} + 1\right]^{\frac{1}{\epsilon-1}}$$

  use relative demand again to substitute $\frac{P_H}{P_L}$

- then: $$g = \frac{\left[L^{\epsilon-1} + (ZH)^{\epsilon-1}\right]^{\frac{1}{\epsilon-1}} - \rho}{\mu\sigma}$$

Technology Transfer: the South

- add a less developed country, South

- same economic structure as the North, but the South is skill-scarce:

$$H^N/L^N > H^S/L^S$$

- no trade in goods or capital

- North innovates, South imitates

- no international patent protection:
  1. new intermediates introduced in the North are immediately copied in the South too (same A_L and A_H)
  2. Northern innovators do not make any profit from the South

Inappropriate Technologies (Acemoglu and Zilibotti, 2001)

- the South lacks the skill endowment required to use optimally technologies designed in the North

- GDP is:

$$Y = \left[(A_L L)^{\frac{\epsilon-1}{\epsilon}} + (Z A_H H)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}$$

  static efficiency requires

$$\frac{\partial Y}{A_L} = \frac{\partial Y}{A_H} \Leftrightarrow \frac{A_H}{A_L} = \left(\frac{ZH}{L} \right)^{\epsilon-1}$$

- the condition is satisfied in North, not in South $(H^N/L^N > H^S/L^S)$

- even if North and South use the same technology, the South is less efficient

Inappropriate Technologies Continued

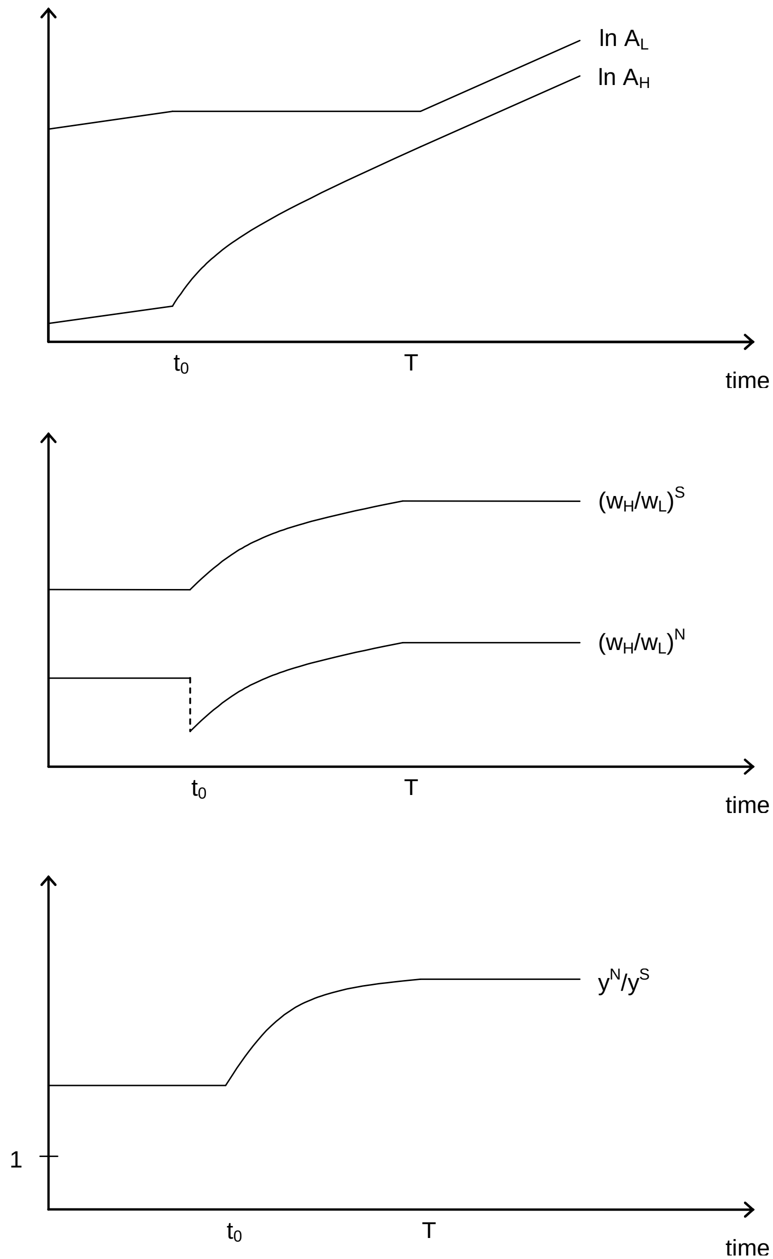
- moreover:

$$\frac{\partial\left(y^N/y^S\right)}{\partial\left(A_H/A_L\right)} > 0$$

  where y is GDP per effective worker

- skill-biased technical change (SBTC) tend to increase cross country income differences

- next picture: effect of SBTC due to $\uparrow H^N/L^N$ when $\epsilon > 2$
  - higher skill premia worldwide (more in South)
  - higher North-South income differences



ln A_L
ln A_H
t_0
T
time
(w_H/w_L)^S
(w_H/w_L)^N
t_0
T
time
y^N/y^S
1
t_0
T
time

Competition and Technology Misallocation

- misallocations between sectors and firms are important for TFP (Banerjee & Duflo 2005, Hsieh & Klenow 2007, Epifani & Gancia 2008)

- if market power varies across firms (e.g., due to political linkages) $\longrightarrow$ technology misallocation

- assume that competition (σ) varies across the two sectors

$$\frac{\pi_H}{\pi_L} = \frac{P_H Z H}{\sigma_H} \frac{\sigma_L}{P_L L}$$

$$\pi_H = \pi_L \;\rightarrow\; \frac{A_H}{A_L} = \left(\frac{\sigma_L}{\sigma_H}\right)^{\epsilon} \left(\frac{ZH}{L}\right)^{\epsilon-1}$$

A_H/A_L not optimal as long as $\sigma_L \neq \sigma_H$
why? high rent sectors attract too much innovation

Trade and Skill-Biased Technical Change

- trade has major effects on the wealth of nation
  even more, when it interacts with technology

- trade may lead to SBTC:
  Acemoglu (2002):
  North-South trade increases $\frac{P_H}{P_L}$ in the North (price effect)
  Epifani & Gancia (2008):
  North-North trade is skill-biased if skilled workers produce more differentiated goods (market size effect)

- in turn, a higher skill premium is more beneficial to skill-abundant countries

Conclusions

- technology is one of the most important factor explaining cross-country income differences

- technolgical change is also responsible for within-country inequality

- models of directed technical change show that studying the “shape” (not just the level) of technology is crucial

- profits/incentives affect on only the level of R&D investment but also the type of innovations in important ways