

# Monetary Policy and Bubbles in a New Keynesian Model with Overlapping Generations

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## Abstract

I develop an extension of the basic New Keynesian model with overlapping generations of finitely-lived agents. In contrast with the standard model, the proposed framework allows for the existence of rational expectations equilibria featuring asset price bubbles. I examine the conditions under which bubbly equilibria may emerge and the implications for the design of monetary policy. Monetary policies that lean against the bubble are shown to be potentially destabilizing, and likely to be dominated by inflation targeting policies.

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Speculative bubbles in asset prices are viewed by many economists and policymakers as an important source of macroeconomic instability, with the bursting of some large bubble often pointed to as a key factor behind many financial crises. A monetary policy that focuses narrowly on inflation and output stability but which neglects the emergence and rapid growth of asset bubbles is often perceived as a potential risk to medium-term macroeconomic and financial stability.<sup>1</sup>

Interestingly, the recurrent reference to bubbles in the policy debate contrasts with their conspicuous absence in modern monetary models. A likely explanation for this seeming anomaly lies in the fact that standard versions of the New Keynesian model, the workhorse framework used in monetary policy analysis, leave no room for the existence of bubbles in equilibrium, and hence for any meaningful *model-based* discussion of their possible interaction with monetary policy.<sup>2</sup>

In the present paper I develop a modified version of the New Keynesian model featuring overlapping generations of finite-lived consumers with retirement.<sup>3</sup> The assumption of an infinite sequence of generations makes it possible for the transversality condition of any *individual* consumer to be satisfied in equilibrium, even in the presence of a bubble that grows at the rate of interest.<sup>4</sup> On the other hand, the assumption of retirement (or, more generally, of an eventual transition to inactivity) can generate an equilibrium rate of interest below the economy's trend growth rate, which is a condition necessary for the size of the bubble to remain bounded relative to the size of the economy. Finally, and in contrast with most models with bubbles found in the literature, the assumption of sticky prices –a key feature of the New Keynesian model– makes monetary policy non-neutral, allowing it to influence the size of the bubble; on the other hand, price stickiness makes it possible for aggregate bubble fluctuations to influence aggregate demand and, hence, output and employment. An appealing feature of the framework developed here is

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<sup>1</sup>See, e.g., Borio and Low (2002) for an early statement of that view. Taylor (2014) points to excessively low interest rates in the 2000s as a factor behind the housing boom that preceded the financial crisis of 2007-2008.

<sup>2</sup>The reason is well known: the equilibrium requirement that the bubble grows at the rate of interest violates the transversality condition of the infinite-lived representative consumer assumed in the New Keynesian model (as well as most macro models). See, e.g., Santos and Woodford (1997).

<sup>3</sup>Other authors have extended the New Keynesian model to incorporate overlapping generations of finite-lived agents into the New Keynesian framework, though none of them has allowed for the existence of bubbles. Piergallini (2006) develops a related model with money in the utility function to analyze the implications of the real balance effect on the stability properties of interest rate rules. Nisticò (2012) discusses the desirability of a systematic monetary policy response to stock price developments in a similar model, but in the absence of bubbles. Del Negro, Giannoni and Patterson (2015) propose a related framework as a possible solution to the "forward guidance puzzle." None of the previous authors allow for retirement in their frameworks. That feature plays a central role in the emergence of asset price bubbles in the model proposed here.

<sup>4</sup>And even though that transversality condition does not hold for the economy as a whole.

that it nests the standard New Keynesian model as a limiting case, when the probability of death and that of retirement approach zero.

After deriving the equations describing the model's equilibrium, I characterize the balanced growth paths consistent with that equilibrium and discuss the conditions under which a non-vanishing bubble may exist along those paths. If the probability of retirement is sufficiently low (relative to the consumer's discount rate), there exists a unique balanced growth path, and it is a bubbleless one (as in the standard model). On the other hand, if the probability of retirement is sufficiently high (but plausibly so), a multiplicity of bubbly balanced growth paths is shown to exist, in addition to a bubbleless one (which always exists).

Once I characterize the existence and potential multiplicity of balanced growth paths –bubbly and bubbleless– I turn to the analysis of the stability properties of those paths and the role of monetary policy in determining those properties.

Several findings of interest emerge from that analysis. First, even in the absence of bubbles, the possibility of an interest rate below the growth rate, when combined with a sufficiently low responsiveness of inflation to the output gap, implies that a kind of "reinforced Taylor principle" is needed in order to guarantee a locally unique equilibrium.

A second finding of interest relates to the possibility of expectations-driven fluctuations in a neighborhood of a balanced growth path. I show that, in contrast with the standard New Keynesian model, the Taylor principle generally fails to guarantee the local uniqueness of the equilibrium. As a result, fluctuations in economic activity and inflation may often arise in association with fluctuations in the size of the bubble, even in the absence of any shocks to fundamentals. An interest rate rule that responds directly to the bubble, if precisely calibrated, may succeed in insulating output and inflation from aggregate bubble fluctuations, without necessarily bursting the bubble or dampening its fluctuations (and possibly amplifying the latter). However, mismeasurement of the bubble or an inaccurate "calibration" of the policy response to its fluctuations may end up destabilizing output and inflation. By way of contrast, I show that a policy that targets inflation directly may attain the same stabilization objectives without the risks associated with direct responses to the bubble.

The paper concludes with some reflections on some of the caveats and limitations of the New Keynesian model developed here as a framework to capture the role played by bubbles as a source

of economic fluctuations.

The rest of the paper is organized as follows. The next section summarizes the related literature. Section 2 describes the basic framework underlying the analysis in the rest of the paper. Section 3 characterizes the economy's balanced growth paths, bubbleless and bubbly. Section 4 analyzes the equilibrium dynamics in a neighborhood of a balanced growth path, and the role of monetary policy in preventing indeterminacy of equilibria. Section 5 provides an example of aggregate fluctuations driven by a (stochastic) bubble, and of the possible consequences of "leaning against the bubble" policies. Section 6 summarizes and concludes.

## 1 Related Literature

Much of the analysis of rational bubbles in general equilibrium found in the literature has been based on real models. An early reference in that category is Tirole (1985), using a conventional overlapping generations (OLG) framework with capital accumulation. A more recent one is Martín and Ventura (2012), who modify the Tirole model by introducing financial frictions that are alleviated by the existence of a bubble.

There is also an extensive literature on bubbles using monetary models with fully flexible prices. In most of those models, including the seminal paper by Samuelson (1958), money itself is the bubbly asset. Asriyan et al. (2016) provide a more recent example, introducing the notion of a nominal bubble. While monetary policy is not always neutral in those models, the mechanism through which its effects are transmitted is very different from that emphasized in standard models with nominal rigidities.

A number of papers have modified the standard New Keynesian model by introducing overlapping generations à la Blanchard-Yaari, though none of them has considered the possible existence of bubbles. Piergallini (2006) develops a related model with money in the utility function to analyze the implications of the real balance effect on the stability properties of interest rate rules. Nisticò (2012) discusses the desirability of a systematic monetary policy response to stock price developments in a similar model, but in the absence of bubbles. Del Negro, Giannoni and Patterson (2015) propose a related framework as a possible solution to the "forward guidance puzzle." None of the previous authors allow for retirement in their frameworks. That feature plays a central

role in the emergence of asset price bubbles in the model proposed here.

Bernanke and Gertler (1999, 2001) analyze the possible gains from "leaning against the wind" monetary policies in a New Keynesian model in which stock prices contain an *ad-hoc* deviation from their fundamental value. The properties of that deviation differ from those of a rational bubble, which cannot exist in their model, which assumes an infinite-lived representative consumer.

In Galí (2014) I carried out a similar analysis of monetary policy rules in a sticky price model in which rational bubbles may exist in equilibrium due to the assumption of an infinite sequence of overlapping generations. While closest in spirit to the present paper, the framework used in that paper differed significantly from the New Keynesian framework in many dimensions. In particular, employment and output were constant in equilibrium, independently of fluctuations in the aggregate bubble, which had only a redistributive effect. On the other hand, the assumption of two-period lived individuals, while convenient, cannot be easily reconciled with the frequency of observed asset boom-bust episodes (not to say with the observed duration of individual prices). By way of contrast, the model developed here displays endogenous fluctuations in output and employment in response to fluctuations in asset price bubbles, and it is consistent with a calibration of the model to a quarterly frequency (as is convention in the business cycle literature). Finally, an additional advantage of the framework developed below is that it nests the standard New Keynesian model as a limiting case.

## 2 A New Keynesian Model with Overlapping Generations

Next I describe the basic framework underlying the analysis in the rest of the paper.

### 2.1 Consumers

I assume an economy with overlapping generations of the "perpetual youth" type, as in Yaari (1965) and Blanchard (1984). The size of the population is constant and normalized to one. Each individual has a constant probability  $\gamma$  of surviving into the following period, independently of his age and economic status ("active" or "retired"). A cohort of size  $1 - \gamma$  is born (in an economic sense) and becomes active each period. Thus, the size in period  $t \geq s$  of a cohort born in period  $s$  is given by  $(1 - \gamma)\gamma^{t-s}$ .

At any point in time, active and retired individuals coexist in the economy. Active individuals

supply labor and manage their own firms, which they set up when they join the economy. I assume that each active individual faces a constant probability  $1 - v$  of permanently losing his job and quitting his entrepreneurial activities. That probability is independent of his age. For convenience, below I refer to that transition as "retirement," though it should be clear that it can be given a broader interpretation.<sup>5</sup>

The previous assumptions imply that the size of the active population (and, hence, the measure of firms) at any point in time is constant and given by  $\alpha \equiv (1 - \gamma)/(1 - v\gamma) \in (0, 1]$ .

A representative consumer from cohort  $s$ , standing in period 0, chooses a consumption plan to maximize expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} (\beta\gamma)^t \log C_{t|s}$$

subject to the sequence of period budget constraints

$$\frac{1}{P_t} \int_0^{\alpha} P_t(i) C_{t|s}(i) di + E_t \{ \Lambda_{t,t+1} Z_{t+1|s} \} = A_{t|s} + W_t N_{t|s} \quad (1)$$

for  $t = 0, 1, 2, \dots$ .  $\beta \equiv 1/(1 + \rho) \in (0, 1)$  is the discount factor.  $C_{t|s} \equiv \left( \alpha^{-\frac{1}{\epsilon}} \int_0^{\alpha} C_{t|s}(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$  is a consumption index, with  $C_{t|s}(i)$  being the quantity purchased of good  $i \in [0, \alpha]$ , at a price  $P_t(i)$ .  $P_t \equiv \left( \alpha^{-1} \int_0^{\alpha} P_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$  is the price index.

Complete markets for state-contingent securities are assumed, with  $Z_{t+1|s}$  denoting the stochastic payoff (expressed in units of the consumption index) generated by a portfolio of securities purchased in period  $t$ , with value given by  $E_t \{ \Lambda_{t,t+1} Z_{t+1|s} \}$ , where  $\Lambda_{t,t+1}$  is the stochastic discount factor for one-period-ahead (real) payoffs. Only individuals who are alive can trade in securities markets.

Variable  $A_{t|s}$  denotes financial wealth at the start of period  $t$ , for a member of cohort  $s \leq t$ . For individuals other than those joining the economy in the current period,  $A_{t|s} = Z_{t|s}/\gamma$ , where the term  $1/\gamma$  captures the additional return on wealth resulting from an annuity contract. As in Blanchard (1984), that contract has the holder receive each period from a (perfectly competitive) insurance firm an annuity payment proportional to his financial wealth, in exchange for transferring

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<sup>5</sup>Gertler (1999) introduces retirement in a similar fashion in a model of social security. More recently, Carvalho et al (2016) have used a version of the Gertler model to analyze the sources of low frequency changes in the equilibrium real rate. Both papers develop *real* models, in contrast to the present one, and do not consider the possibility of bubbles.

the latter to the insurance firm upon death.<sup>6</sup>

Variable  $W_t$  denotes the (real) wage per hour, and  $N_{t|s}$  is the number of individual work hours. Both the wage and work hours are taken as given by each worker and assumed to be common to all active individuals, i.e.  $N_{t|s} = N_t/\alpha$ , where  $N_t$  is aggregate employment.<sup>7</sup> Note that  $N_{t|s} = 0$  for retired individuals.

Finally, I assume a solvency constraint of the form  $\lim_{T \rightarrow \infty} \gamma^T E_t \{ \Lambda_{t,t+T} A_{t+T|s} \} \geq 0$  for all  $t$ , where  $\Lambda_{t,t+T}$  is determined recursively by  $\Lambda_{t,t+T} = \Lambda_{t,t+T-1} \Lambda_{t+T-1,t+T}$ .<sup>8</sup>

The problem above yields a set of optimal demand functions

$$C_{t|s}(i) = \frac{1}{\alpha} \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_{t|s} \quad (2)$$

for all  $i \in [0, \alpha]$ , which in turn imply  $\int_0^\alpha P_{t|s}(i) C_{t|s}(i) di = P_t C_{t|s}$ . Thus we can rewrite the period budget constraint as:

$$C_{t|s} + \gamma E_t \{ \Lambda_{t,t+1} A_{t+1|s} \} = A_{t|s} + W_t N_{t|s} \quad (3)$$

The consumer's optimal plan must satisfy the optimality condition<sup>9</sup>

$$\Lambda_{t,t+1} = \beta \frac{C_{t|s}}{C_{t+1|s}} \quad (4)$$

and the transversality condition

$$\lim_{T \rightarrow \infty} \gamma^T E_t \{ \Lambda_{t,t+T} A_{t+T|s} \} = 0 \quad (5)$$

with (4) holding for all possible states of nature (conditional on the individual remaining alive in  $t + 1$ ).

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<sup>6</sup>Thus, individuals who hold negative assets will pay an annuity fee to the insurance company. The latter absorbs the debt in case of death. The insurance arrangement can also be replicated through securities markets. In that case the individual will purchase a portfolio that generates a random payoff  $A_{t+1|s}$  if he remains alive, 0 otherwise. The value of that payoff will be given by  $E_t \{ \Lambda_{t,t+1} \gamma A_{t+1|s} \}$  which is equivalent to the formulation in the main text, given that  $A_{t|s} = Z_{t|s}/\gamma$ .

<sup>7</sup>By not including hours of work in the utility function I effectively eliminate any wealth effects that would generate systematic counterfactual differences in the quantity of labor supplied by active individuals across age groups. Alternatively one may assume preferences that rule out those wealth effects, but at the cost of rendering the analysis below less tractable.

<sup>8</sup>Note that  $(\Lambda\gamma)^{-1}$  is the "effective" interest rate paid by a borrower in the steady state. The solvency constraint thus has the usual interpretation of a no-Ponzi game condition.

<sup>9</sup>Note that in the optimality condition the probability of remaining alive  $\gamma$  and the extra return  $1/\gamma$  resulting from the annuity contract cancel each other. Complete markets guarantee the same consumption growth rate between two different periods for all consumers alive in the two periods.

The details of wage setting are not central to the main point of the paper. For convenience, I assume an ad-hoc wage schedule linking the productivity-adjusted real wage to average work hours:

$$\mathcal{W}_t = \left( \frac{N_t}{\alpha} \right)^\varphi \quad (6)$$

where  $N_t \equiv \int_0^\alpha N_t(i) di$  denotes aggregate work hours.

## 2.2 Firms

Each individual is endowed with the know-how to produce a differentiated good, and sets up a firm with that purpose when he joins the economy. That firm remains operative until its founder retires or dies, whatever comes first.<sup>10</sup> All firms have an identical technology, represented by the linear production function

$$Y_t(i) = \Gamma^t N_t(i) \quad (7)$$

where  $Y_t(i)$  and  $N_t(i)$  denote output and employment for firm  $i \in [0, \alpha]$ , respectively, and  $\Gamma \equiv 1 + g \geq 1$  denotes the (gross) rate of productivity growth. Individuals cannot work at their own firms, and must hire instead labor services provided by others.<sup>11</sup>

Aggregation of (2) across consumers yields the demand schedule facing each firm

$$C_t(i) = \frac{1}{\alpha} \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t$$

where  $C_t \equiv (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} C_{t|s}$  is aggregate consumption. Each firm takes as given the aggregate price level  $P_t$  and aggregate consumption  $C_t$ .

As in Calvo (1983), each *incumbent* firm resets the price of its good with probability  $1 - \theta_I$  in any given period, and keeps its unchanged with probability  $\theta_I$ . Those probabilities are independent of the time elapsed since the last price adjustment. On the other hand, I assume that a fraction  $1 - \theta_N$  of newly created firms sets its price optimally at birth, while the remaining fraction  $\theta_N$  set a price equal to the economy's average price in the previous period.<sup>12</sup> Letting  $\theta \equiv v\gamma\theta_I + (1 - v\gamma)\theta_N$

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<sup>10</sup>The assumption of finite-lived firms (or more generally, for firms whose dividends shrink relative to the size of the economy) is needed in order for bubbles to exist in equilibrium. By equating the probability of a firm's survival to that of its owner remaining alive and active I effectively equate the rate at which dividends and labor income are discounted, which simplifies considerably the analysis below.

<sup>11</sup>I assume that each firm newly set up in any given period inherits (through random assignment) the index of an exiting firm.

<sup>12</sup>Alternatively, a fraction  $\theta_N$  "inherit" the price in the previous period for the good they replace. In either case I implicitly assume a transfer system which equalizes the wealth across members of the new cohort.



the aggregate price dynamics are described by the equation

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}$$

where  $P_t^*$  is the price set in period  $t$  by firms reoptimizing their price.<sup>13</sup> A log-linear approximation of the previous difference equation around the zero inflation equilibrium yields (letting lower case letters denote the logs of the original variables):

$$p_t = \theta p_{t-1} + (1-\theta)p_t^* \quad (8)$$

i.e. the current price level is a weighted average of last period's price level and the newly set price, all in logs, with the weights given by the fraction of firms that do not and do adjust prices, respectively.

In both environments, a firm adjusting its price in period  $t$  will choose the price  $P_t^*$  that maximizes

$$\max_{P_t^*} \sum_{k=0}^{\infty} (v\gamma\theta_I)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left( \frac{P_t^*}{P_{t+k}} - \mathcal{W}_{t+k} \right) \right\}$$

subject to the sequence of demand constraints

$$Y_{t+k|t} = \frac{1}{\alpha} \left( \frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} C_{t+k} \quad (9)$$

for  $k = 0, 1, 2, \dots$  where  $Y_{t+k|t}$  denotes output in period  $t+k$  for a firm that last reset its price in period  $t$  and  $\mathcal{W}_t \equiv W_t/\Gamma^t$  is the productivity-adjusted real wage (i.e. the real marginal cost).<sup>14</sup>

The optimality condition associated with the problem above takes the form

$$\sum_{k=0}^{\infty} (v\gamma\theta_I)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left( \frac{P_t^*}{P_{t+k}} - \mathcal{M}\mathcal{W}_{t+k} \right) \right\} = 0 \quad (10)$$

where  $\mathcal{M} \equiv \frac{\epsilon}{\epsilon-1}$  is the optimal markup under flexible prices.

A first-order Taylor expansion of (10) around the zero inflation steady state yields, after some manipulation:

$$p_t^* = \mu + (1 - \Lambda\Gamma v\gamma\theta_I) \sum_{k=0}^{\infty} (\Lambda\Gamma v\gamma\theta_I)^k E_t \{\psi_{t+k}\} \quad (11)$$

where  $\psi_t \equiv \log P_t \mathcal{W}_t$  is the (log) marginal cost,  $\mu \equiv \log \mathcal{M}$  and  $\Lambda \equiv 1/(1+r)$  is the steady state stochastic discount factor. Note that along a zero inflation balanced growth path,  $w_t = w = -\mu$ .

<sup>13</sup>Note that the price is common to all those firms, since they face an identical problem.

<sup>14</sup>The firm's demand schedule can be derived by aggregating (2) across cohorts.

Throughout I maintain the assumption that  $\Lambda\Gamma v\gamma \in [0, 1)$ , which guarantees that the infinite sum in (10) converges in a neighborhood of the zero inflation balanced growth path.

Letting  $\mu_t \equiv p_t - \psi_t$  denote the average (log) price markup, and combining (8) (or (??)) and (11) yields the inflation equation:

$$\pi_t = \Phi E_t\{\pi_{t+1}\} - \lambda(\mu_t - \mu) \quad (12)$$

where  $\pi_t \equiv p_t - p_{t-1}$  denotes inflation,  $\Phi \equiv \Lambda\Gamma v\gamma\theta_I/\theta$  and  $\lambda \equiv (1 - \theta)(1 - \Lambda\Gamma v\gamma\theta_I)/\theta > 0$ .<sup>15</sup>

Wage schedule (6), together with the assumption of a constant gross markup  $\mathcal{M}$  under flexible prices, implies a natural level of output given by  $Y_t^n = \Gamma^t \mathcal{Y}$ , where  $\mathcal{Y} \equiv \alpha \mathcal{M}^{-\frac{1}{\varphi}}$ . Log-linearizing (6), and combining the resulting expressions with (12) we obtain a version of the New Keynesian Phillips curve

$$\pi_t = \Phi E_t\{\pi_{t+1}\} + \kappa \hat{y}_t \quad (13)$$

where  $\kappa \equiv \lambda\varphi$ , and  $\hat{y}_t \equiv \log(Y_t/Y_t^n)$  is the output gap. Note that, in contrast with the standard New Keynesian model, the coefficient on expected inflation is not pinned down by the consumer's discount factor. Instead it depends on demographic parameters ( $v$  and  $\gamma$ ), the relative degree of price stickiness of incumbent firms ( $\theta_I/\theta$ ) and the gap between the real interest rate and the growth rate along a balanced growth path (as captured by  $\Lambda\Gamma$ ), all of which determine the effective "forward-lookingness" of inflation.

## 2.3 Asset Markets

In addition to a complete set of state-contingent securities, I assume the existence of markets for a number of specific assets, whose prices and returns must satisfy some equilibrium conditions. In particular, the yield  $i_t$  on a one-period nominally riskless bond purchased in period  $t$  must satisfy<sup>16</sup>

$$1 = (1 + i_t) E_t \left\{ \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} \right\} \quad (14)$$

thus implying that the relation  $\Lambda \equiv 1/(1 + r)$  between the discount factor and the real return on the riskless nominal bond ( $r$ ) will hold along a perfect foresight balanced growth path.

<sup>15</sup>Note that in the standard model with a representative consumer,  $\Phi \equiv \beta$  and  $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$  which correspond to the limit of the expressions for those coefficients as  $v\gamma \rightarrow 1$  under the two environments, and given that  $\Lambda\Gamma = \beta$  along a balanced growth path of the representative consumer economy.

<sup>16</sup>Note also that in the asset pricing equations the probability of remaining alive  $\gamma$  and the extra return  $1/\gamma$  resulting from the annuity contract cancel each other.

Stocks in individual firms trade at a price  $Q_t^F(i)$ , for  $i \in [0, \alpha]$ , which must satisfy the asset pricing equation:

$$Q_t^F(i) = D_t(i) + v\gamma E_t \{ \Lambda_{t,t+1} Q_{t+1}^F(i) \} \quad (15)$$

where  $D_t(i) \equiv Y_t(i) \left( \frac{P_t(i)}{P_t} - W_t \right)$  denotes firm  $i$ 's dividends, and  $v\gamma$  is the probability that firm  $i$  survives into next period. Solving (15) forward under the assumption that  $\lim_{k \rightarrow \infty} (v\gamma)^k E_t \{ \Lambda_{t,t+k} Q_{t+k}^F(i) \} = 0$ , and aggregating across firms:

$$\begin{aligned} Q_t^F &\equiv \int_0^\alpha Q_t^F(i) di \\ &= \sum_{k=0}^{\infty} (v\gamma)^k E_t \{ \Lambda_{t,t+k} D_{t+k} \} \end{aligned} \quad (16)$$

where  $D_t \equiv \int_0^\alpha D_t(i) di$  denotes aggregate dividends. Note that firms' death makes it possible for the aggregate value of firms to be finite even if the interest rate is below the economy's growth rate, as long as  $\Lambda \Gamma v \gamma < 1$ .

Much of the analysis below focuses on intrinsically worthless assets i.e. assets generating no dividend, pecuniary or not.<sup>17</sup> Let  $Q_t^B(j)$  denote the price of one such asset. In equilibrium that price must satisfy the condition

$$Q_t^B(j) = E_t \{ \Lambda_{t,t+1} Q_{t+1}^B(j) \} \quad (17)$$

as well as the non-negativity constraint  $Q_t^B(j) \geq 0$  (given free disposal), for all  $t$ .

Let  $Q_t^B$  denote the aggregate value of bubbly assets in period  $t$ . In equilibrium, that variable evolves over time according to the following two equations:

$$Q_t^B = U_t + B_t \quad (18)$$

$$Q_t^B = E_t \{ \Lambda_{t,t+1} B_{t+1} \} \quad (19)$$

where  $U_t \equiv Q_{t|t}^B \geq 0$  is the value of a new bubble introduced by cohort  $t$  at birth,<sup>18</sup> and  $B_t \equiv \sum_{s=-\infty}^{t-1} Q_{t|s}^B \geq 0$  is the aggregate value in period  $t$  of bubbly assets available for trade in period  $t-1$ , with  $Q_{t|s}^B$  denoting the period  $t$  value of bubbly assets introduced in period  $s \leq t$ . Note that the introduction of new bubbly assets by incoming cohorts makes it possible for an aggregate

<sup>17</sup>In Jean Tirole's words, pure bubbly assets are "best thought of as pieces of paper."

<sup>18</sup>Think of pieces of paper of a cohort-specific color or stamped with the birth year of their creators.

bubble to re-emerge after a hypothetical collapse, thus overcoming a common criticism of early rational bubble models. A similar environment with bubble creation was first introduced and analyzed in Martín and Ventura (2012) in the context of an overlapping generations model with financial frictions.<sup>19</sup>

Note that in the previous environment, the initial financial wealth of a member of a cohort born in period  $t$  is given by:

$$A_{t|t} = Q_{t|t}^F + U_t/(1 - \gamma)$$

where  $Q_{t|t}^F$  is the value in period  $t$  of a newly created firm.

## 2.4 Monetary Policy

Unless otherwise noted, the central bank is assumed to follow a simple interest rate rule of the form

$$\widehat{i}_t = \phi_\pi \pi_t + \phi_q \widehat{q}_t^B \quad (20)$$

where  $\widehat{i}_t \equiv \log[(1 + i_t)/(1 + r)]$  and  $q_t^B \equiv Q_t^B/(\Gamma^t \mathcal{Y})$  is the size of the aggregate bubble normalized by trend output, with  $\widehat{q}_t^B \equiv q_t^B - q^B$  denoting the deviation from its value along a balanced growth path. Note that the previous rule is consistent with a zero inflation target. In what follows, I assume the central bank takes  $r$  and  $q^B$  as given, as determined by the analysis below.

## 2.5 Market Clearing

Goods market clearing requires  $Y_t(i) = (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} C_{t|s}(i)$  for all  $i \in [0, \alpha]$ . Letting  $Y_t \equiv \left( \alpha^{-\frac{1}{\epsilon}} \int_0^\alpha Y_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$  denote aggregate output, we have:

$$\begin{aligned} Y_t &= (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} C_{t|s} \\ &= C_t \end{aligned}$$

Note also that in equilibrium

$$\begin{aligned} N_t &= \int_0^\alpha N_t(i) di \\ &= \Delta_t^p \mathcal{Y}_t \\ &\simeq \mathcal{Y}_t \end{aligned}$$

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<sup>19</sup>The bubble introduced by each individual can be interpreted as being attached to the stock of his firm and hence to burst whenever the firm stops operating (i.e. with probability  $1 - v\gamma$ ).

where  $\mathcal{Y}_t \equiv Y_t/\Gamma^t$  is aggregate output normalized by productivity and  $\Delta_t^p \equiv \frac{1}{\alpha} \int_0^\alpha (P_t(i)/P_t)^{-\epsilon} di \simeq 1$  is an index of relative price distortions, which equals one up to a first-order approximation.

Assuming that all securities other than stocks and bubbly assets are in zero net supply, asset market clearing requires

$$(1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} A_{t|s} = Q_t^F + Q_t^B$$

Next I characterize the economy's perfect foresight, zero inflation balanced growth paths.

### 3 Bubbles and Balanced Growth Paths

In a perfect foresight balanced growth (henceforth, BGP) the discount factor is constant and satisfies  $\Lambda = 1/(1+r)$ , as implied by (14), and where  $r$  is the real interest rate along a BGP. Note also that  $\mathcal{W} = 1/\mathcal{M}$  in the zero inflation BGP. Combined with (6) the previous condition implies:

$$\frac{1}{\mathcal{M}} = \left( \frac{\mathcal{Y}_t}{\alpha} \right)^\varphi$$

Accordingly, output along the BGP, is given by  $Y_t^{BGP} = \Gamma^t \mathcal{Y}$ , which coincides with the natural level of output, as derived above.

Next I describe how aggregate consumption is determined. Details of the derivation can be found in the Appendix.

Let  $\mathcal{C}_j$  and  $\mathcal{A}_j^i$  denote, respectively, consumption and financial wealth along a BGP for an individual aged  $j$ , *normalized by productivity*, i.e. superindex  $i \in \{a, r\}$  denotes his status as active or retired. The intertemporal budget constraint for a consumer born in period  $s$  and who remains active at time  $t$ , derived by solving (1) forward, can be evaluated at a BGP as:

$$\sum_{k=0}^{\infty} (\Lambda \Gamma \gamma)^k \mathcal{C}_{t+k-s} = \mathcal{A}_{t-s}^a + \frac{1}{1 - \Lambda \Gamma v \gamma} \left( \frac{\mathcal{W}N}{\alpha} \right)$$

where  $N$  and  $\mathcal{W}$  denote aggregate hours and the wage (the latter normalized by productivity) along the BGP.

Using the fact that  $\mathcal{C}_{t+k-s} = [\beta(1+r)/\Gamma]^k \mathcal{C}_{t-s}$  –as implied by (4) evaluated at the BGP – the following consumption function can be obtained for an active individual aged  $j$ :

$$\mathcal{C}_j = (1 - \beta\gamma) \left[ \mathcal{A}_j^a + \frac{1}{1 - \Lambda \Gamma v \gamma} \left( \frac{\mathcal{W}N}{\alpha} \right) \right] \quad (21)$$

Thus, consumption for an active individual is proportional to the sum of his financial wealth,  $\mathcal{A}_j^a$ , and his current and future labor income (properly discounted),  $\mathcal{W}N/[\alpha(1 - \Lambda\Gamma v\gamma)]$ .

The corresponding consumption function for a retired individual is given by:

$$\mathcal{C}_j = (1 - \beta\gamma)\mathcal{A}_j^r \quad (22)$$

Aggregating over all individuals, imposing the asset market clearing condition  $\mathcal{A} = \mathcal{Q}^F + \mathcal{Q}^B$  (with these three variables now normalized by productivity) and using the fact that  $\mathcal{Q}^F = \mathcal{D}/(1 - \Lambda\Gamma v\gamma)$  and  $\mathcal{Y} = \mathcal{W}N + \mathcal{D}$  along a BGP (with  $\mathcal{D} \equiv (1 - 1/\mathcal{M})\mathcal{Y}$  denoting aggregate profits normalized by productivity), we obtain:

$$\mathcal{C} = (1 - \beta\gamma) \left[ \mathcal{Q}^B + \frac{1}{1 - \Lambda\Gamma v\gamma} \mathcal{Y} \right] \quad (23)$$

which can be interpreted as an aggregate consumption function along the BGP. Finally, goods market clearing requires that  $\mathcal{C} = \mathcal{Y}$  thus implying the following equation relating the bubble-output ratio  $q^B \equiv \mathcal{Q}^B/\mathcal{Y}$  and the discount factor \*

$$1 = (1 - \beta\gamma) \left[ q^B + \frac{1}{1 - \Lambda\Gamma v\gamma} \right] \quad (24)$$

Next I turn to the analysis of "bubbleless" and "bubbly" BGPs.

### 3.1 Bubbleless Balanced Growth Paths

Consider first a "bubbleless" BGP, with  $q^B = 0$ . Imposing that condition in (24) implies

$$\Lambda\Gamma v = \beta$$

or, equivalently,

$$r = (1 + \rho)(1 + g)v - 1$$

Note that the real interest rate along a BGP is increasing in both  $v$  and  $g$ . The reason is that an increase in either of those variables raises desired consumption by increasing the expected present discounted value of future income for currently active individuals. In order for the goods market to clear, an increase in the interest rate is called for.

When  $v = 1$ , the real interest rate is given (approximately) by the discount rate plus the growth rate, i.e.  $r \simeq \rho + g$ , as in the standard model (with log utility, as assumed here).

Note also that an increase in the expected lifetime, as indexed by  $\gamma$ , does not have an independent effect on the real interest rate along the bubbleless BGP. The reason is that, when  $\Lambda\Gamma v = \beta$ , such a change increases in the same proportion the present value of consumption and that of income for any given real rate, making an adjustment in the latter unnecessary.<sup>20</sup>

Finally, note for future reference that in the bubbleless BGP considered here the real interest rate  $r$  is lower than the growth rate  $g$  (i.e.  $\Lambda\Gamma > 1$ ) if and only if  $v < \beta$ .

### 3.2 Bubbly Balanced Growth Paths

Next I consider the possibility of a BGP with an aggregate bubble growing at the same rate as output, thus implying a constant bubble-output ratio  $q^B > 0$ .

Letting  $q_t^B \equiv Q_t^B/(\Gamma^t \mathcal{Y})$ ,  $b_t \equiv B_t/(\Gamma^t \mathcal{Y})$  and  $u_t \equiv U_t/(\Gamma^t \mathcal{Y})$ , note first that (19) can be rewritten as:

$$\begin{aligned} q_t^B &= E_t\{\Lambda_{t,t+1}\Gamma b_{t+1}\} \\ &= E_t\{\Lambda_{t,t+1}\Gamma(q_{t+1}^B - u_{t+1})\} \end{aligned} \quad (25)$$

Along a bubbly BGP we must have  $q_t^B = q^B > 0$ , and  $u_t = u \geq 0$  for all  $t$ . It follows from (24) and (25) that:

$$\begin{aligned} q^B &= \frac{\gamma(\beta - \Lambda\Gamma v)}{(1 - \beta\gamma)(1 - \Lambda\Gamma v\gamma)} \\ u &= \left(1 - \frac{1}{\Lambda\Gamma}\right) q^B \end{aligned} \quad (26)$$

Thus, the existence of a BGP with a positive bubble,  $q^B > 0$ , requires that

$$\Lambda\Gamma v < \beta$$

On the other hand the non-negativity constraint on newly created bubbles  $u \geq 0$  requires:

$$\Lambda\Gamma \geq 1$$

Accordingly, a necessary and sufficient condition for the existence of a bubbly BGP is given by

$$v < \beta \quad (27)$$

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<sup>20</sup>The independence of the steady state real interest rate from  $\gamma$  is a consequence of the log utility specification assumed here. That property is not critical from the viewpoint of the present paper, since there are other factors (the probability of retirement, in particular), that can drive real interest rate to values consistent with the presence of bubbles.

If the previous condition is satisfied, there exists continuum of bubbly BGPs  $\{q^B, u\}$  indexed by  $r \in (\Gamma v/\beta - 1, \Gamma - 1]$ . Note that the condition for the existence of bubbly BGPs is equivalent to the real interest rate being less than the growth rate in the bubbleless BGP.

It can be easily checked that  $q^B$  is increasing in  $r$ , with  $\lim_{r \rightarrow g} q^B = \frac{\gamma(\beta-v)}{(1-\beta\gamma)(1-v\gamma)} \equiv q_{\max}^B$ . Note also that  $\partial q_{\max}^B / \partial v < 0$ , i.e. the upper bound on the size of the bubble is decreasing in  $v$  over the range  $v \in [0, \beta]$ , and converges to zero as  $v \rightarrow \beta$ .

One particular such bubbly BGP has a constant supply of bubbly assets, i.e.  $u = 0$ . Note that in that case

$$\Lambda\Gamma = 1$$

or, equivalently,

$$r = g$$

with the implied bubble size given by  $q^B = q_{\max}^B$ . Along that BGP any existing bubble will be growing at the same rate as the economy. By contrast, along a bubbly BGP with bubble creation,  $r < g$  implies that the size of the *aggregate* pre-existing bubble will be shrinking over time relative to the size of the economy, with newly created bubbles filling up the gap so that the size of the aggregate bubble relative to the size of the economy remains unchanged.

Summing up, one can distinguish two regions of the parameter space relevant for the possible existence of bubbly BGPs:

(i)  $\beta \leq v \leq 1$ . In this case, the BGP is unique and bubbleless and associated with a real interest rate given by  $r = \Gamma v/\beta - 1 > g$ .

(ii)  $0 < v < \beta$ . In this case multiple BGPs coexist. One of them is bubbleless, with  $r = \Gamma v/\beta - 1 < g$ . In addition, there exists a continuum of bubbly BGPs, indexed by the real interest rate  $r \in (\Gamma v/\beta - 1, \Gamma - 1]$ , and associated with a bubble size (relative to output)  $q^B \in (0, q_{\max}^B]$ , given by (26).

Figure 1 summarizes graphically the two regions with their associated BGPs.



### 3.3 A Brief Detour: Bubbly Equilibria and Transversality Conditions

Equilibria with bubbles on assets in positive net supply can be ruled out in an economy with an infinite lived representative consumer.<sup>21</sup> In that economy, any positive net supply of that asset must be necessarily held by the representative consumer, implying  $\lim_{T \rightarrow \infty} E_t \{ \Lambda_{t,t+T} A_{t+T} \} \geq \lim_{T \rightarrow \infty} E_t \{ \Lambda_{t,t+T} Q_{t+T}^B(j) \}$ . Given that the bubble component of any asset must satisfy  $\lim_{T \rightarrow \infty} E_t \{ \Lambda_{t,t+T} Q_{t+T}^B(j) \} \geq Q_t^B(j)$ , it follows that  $\lim_{T \rightarrow \infty} E_t \{ \Lambda_{t,t+T} A_{t+T} \} \geq Q_t^B(j)$ . But the consumer's transversality condition requires that  $\lim_{T \rightarrow \infty} E_t \{ \Lambda_{t,t+T} A_{t+T} \} = 0$ . Given that free disposal requires that  $Q_t^B(j) \geq 0$ , it follows that  $Q_t^B(j) = 0$  for all  $t$ .

Note that the previous reasoning cannot be applied to an overlapping generations economy like the one developed above. The reason is that in that case it is no longer true that the positive net supply of any bubbly asset must be held (asymptotically) by any individual agent, since it can always be passed on to future cohorts (and it will in equilibrium).

In fact, it is easy to check that in the overlapping generations model above the individual transversality condition is satisfied along any BGP, bubbly or bubbleless. As shown in the appendix, for an individual born in period  $s \leq t$  it must be the case that along any BGP

$$\begin{aligned} \lim_{T \rightarrow \infty} \gamma^T E_t \{ \Lambda_{t,t+T} A_{t+T|s} \} &= \lim_{T \rightarrow \infty} \gamma^T [v^T E_t \{ \Lambda_{t,t+T} A_{t+T|s}^a \} + (1 - v^T) E_t \{ \Lambda_{t,t+T} A_{t+T|s}^r \}] \\ &= \lim_{T \rightarrow \infty} (\beta\gamma)^T \left[ A_{s|s}^a + \frac{W_s N}{1 - \Lambda\Gamma v\gamma} \left( 1 - \left( \frac{\Lambda\Gamma v}{\beta} \right)^T \right) \right] \\ &= 0 \end{aligned}$$

implying that the transversality condition is satisfied along any admissible BGP. It is straightforward to show that this will be the case along any equilibrium that remains in a neighborhood of a BGP, of the kind analyzed below.

### 3.4 Plausibility of Bubbly BGPs: Some Rough Numbers

Next I discuss plausible settings for the parameters of the model involved in the above characterization of the BGPs. To calibrate  $\gamma$  I use the expected lifetime at age 16, which is 63.2 years in the

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<sup>21</sup>See, e.g. Santos and Woodford (1997) for a discussion of the conditions under which rational bubbles can be ruled out in equilibrium.

U.S., and set  $\gamma = 1 - \frac{1}{4 \cdot (62.3)} \simeq 0.996$ . I use the average employment ratio (relative to population aged 16 and over), which is (roughly) 0.6 on average over the period 1960-2016 as a proxy for  $\alpha$ . Conditional on the previous settings for  $\alpha$  and  $\gamma$  one can derive  $v \simeq 0.9973$ . Thus, the analysis above implies that the existence of bubbly balanced growth paths requires that  $\beta > 0.9973$ .

Unfortunately, the latter condition cannot be verified easily since that parameter is not identified by the above model once the existence of bubbles is allowed for. This is in contrast with the standard representative agent model, for which there is a tight connection between the discount rate and the real interest rate along a BGP.<sup>22</sup> On the other hand, casual introspection suggests that a discount factor of about 0.9 applied to utility 10 years from today (as implied by the lower bound  $\beta = 0.9973$ ) falls within the range of plausibility.

Alternatively, one may examine directly the relation between the average real interest rate,  $r$ , and the average growth rate of output,  $g$ , two observable variables. As discussed above, the existence of bubbly BGPs requires that  $r \leq g$ . Using data on 3-month Treasury bills, GDP deflator and (per capita) GDP, the average values for those variables in the U.S. over the period 1960Q1-2015Q4 are  $r = 1.4\% \div 4 = 0.35\%$  and  $g = 1.6\% \div 4 = 0.4\%$  (or, equivalently,  $\Lambda = 0.9965$  and  $\Gamma = 1.004$ ), values which satisfy the inequality condition necessary for the existence of bubbles. Note also that the above calibration implies  $\Lambda\Gamma v\gamma \simeq 0.9939$ , thus satisfying the condition for a well defined intertemporal budget constraint.

For some of the quantitative analyses below I set the discount factor to be  $\beta = 0.998$ . This is admittedly, an arbitrary choice, but it is consistent with the existence of bubbly BGPs, with associated real interest rates given by the interval  $[0.003348, 0.004]$ .

## 4 Bubbles, Monetary Policy and Equilibrium Fluctuations

Having characterized the BGPs of the model economy, in the present section I shift the focus to the analysis of the equilibrium dynamics in a neighborhood of a given BGP. In particular, I am interested in determining the conditions under which equilibrium fluctuations unrelated to fundamental shocks may emerge, as well as the role that variations in the size of the aggregate bubble and the nature of monetary policy may play in such fluctuations.

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<sup>22</sup>Note that  $\beta = 0.9983$  implies a discount factor of 0.9342 applied to utility 10 years from today, which seems entirely plausible.

As in the standard analysis of the New Keynesian model with a representative agent, I restrict myself to equilibria that remain in a neighborhood of a BGP, with the local equilibrium dynamics being approximated using the log-linearized equilibrium conditions.<sup>23</sup> I leave the analysis of the global equilibrium dynamics –including the possibility of switches between BGPs, the existence of a zero lower bound on interest rates, and other nonlinearities– to future research. Secondly, in analyzing the model’s equilibrium I ignore the existence of fundamental shocks, and focus instead on the possibility of fluctuations driven by self-fulfilling expectations and on the role of bubbles as a source of those fluctuations.<sup>24</sup>

I start by deriving the log-linearized equilibrium conditions around a BGP. In contrast with the New Keynesian model with a representative agent, the individual consumer’s Euler equation and the goods market clearing condition are no longer sufficient to derive an equilibrium relation determining aggregate output as a function of interest rates (i.e. the so-called dynamic IS equation).<sup>25</sup> Instead, the derivation of such a relation requires solving for an aggregate consumption function, by aggregating the individual consumption functions obtained by combining the consumer’s Euler equation and intertemporal budget constraint. Since no exact representation exists for the individual consumption function, I proceed by deriving a log-linearized approximation of that function around a perfect foresight balanced growth path. Then I aggregate the individual consumption functions to obtain an (approximate) *aggregate* consumption function. See the Appendix for detailed derivations.

The resulting representation of the equilibrium dynamics takes a very simple form, involving only a few easily interpretable equations, as shown next.

Letting  $\hat{c}_t \equiv \log(C_t/\Gamma^t \mathcal{C})$  denote log deviation of aggregate consumption from its value along a BGP, the goods market clearing condition can be written as:

$$\hat{y}_t = \hat{c}_t \tag{28}$$

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<sup>23</sup>See, e.g., Woodford (2003) or Galí (2015).

<sup>24</sup>As the analysis of the equilibrium dynamics below will make clear, in the absence of bubbles and/or multiplicity of equilibria the economy’s behavior in response to fundamental shocks should involve no significant differences relative to that of the standard New Keynesian model with a representative agent.

<sup>25</sup>Formally, one can use the definition of aggregate consumption and the individual Euler equations to derive the aggregate relation:

$$E_t\{\hat{c}_{t+1}\} = \gamma \left( \hat{c}_t + \hat{i}_t - E_t\{\pi_{t+1}\} \right) + (1 - \gamma) E_t\{\hat{c}_{t+1|t+1}\}$$

where  $E_t\{\hat{c}_{t+1|t+1}\}$  cannot be immediately as a function of  $\hat{c}_t$ .

As shown in the appendix, the aggregate consumption function can be written, up to a first order approximation, as follows:

$$\widehat{c}_t = (1 - \beta\gamma)(\widehat{q}_t^B + \widehat{x}_t) \quad (29)$$

where  $\widehat{q}_t^B \equiv q_t^B - q^B$  and

$$\widehat{x}_t \equiv \sum_{k=0}^{\infty} (\Lambda\Gamma v\gamma)^k E_t\{\widehat{y}_{t+k}\} - \frac{\Lambda\Gamma v\gamma}{1 - \Lambda\Gamma v\gamma} \sum_{k=0}^{\infty} (\Lambda\Gamma v\gamma)^k E_t\{\widehat{r}_{t+k}\} \quad (30)$$

is the *non-bubbly* (or *fundamental*) component of aggregate wealth (i.e. current and future discounted income), with  $\widehat{y}_t \equiv \log(Y_t/\Gamma^t\mathcal{Y})$  denoting the output gap and  $\widehat{r}_t = \widehat{i}_t - E_t\{\pi_{t+1}\}$  the real interest rate, all expressed in log deviations from their values along a BGP. Note that  $\widehat{x}_t$  can also be rewritten in recursive form as:

$$\widehat{x}_t = \Lambda\Gamma v\gamma E_t\{\widehat{x}_{t+1}\} + \widehat{y}_t - \frac{\Lambda\Gamma v\gamma}{1 - \Lambda\Gamma v\gamma} (\widehat{i}_t - E_t\{\pi_{t+1}\}) \quad (31)$$

Log-linearization of (25) around a BGP yields the equations describing fluctuations in the aggregate bubble.

$$\widehat{q}_t^B = \Lambda\Gamma E_t\{\widehat{b}_{t+1}\} - q^B(\widehat{i}_t - E_t\{\pi_{t+1}\}) \quad (32)$$

$$\widehat{q}_t^B = \widehat{b}_t + \widehat{u}_t \quad (33)$$

where  $q^B = \frac{\gamma(\beta - \Lambda\Gamma v)}{(1 - \beta\gamma)(1 - \Lambda\Gamma v\gamma)} \geq 0$  (as derived above) and  $\widehat{u}_t \equiv u_t - u$ .

The New Keynesian Phillips curve derived above and given by

$$\pi_t = \Phi E_t\{\pi_{t+1}\} + \kappa\widehat{y}_t \quad (34)$$

completes the description of the *non-policy block* of the system of difference equations describing the model's equilibrium in a neighborhood of a BGP, where the latter is defined by a pair  $(q^B, \Lambda)$  satisfying the BGP conditions derived in the previous section. The model is then closed by a description of monetary policy, as given by (20).

Before I analyze the solutions to the above system of difference equations, I make a brief digression regarding the size of the bubble effects on output implied by the model. As made clear by (29), consumption moves in proportion to fluctuations in total wealth, bubbly and non-bubbly. Through this channel, a rise in the size of the bubble leads, *ceteris paribus*, to an increase in aggregate demand and output. Note, however, that the marginal propensity to consume out of

wealth,  $1 - \beta\gamma$ , is likely to be very small for any plausible values of the discount factor and the survival rate consistent with the existence of bubbles. Given the requirement that  $v < \beta$ , an upper bound to that marginal propensity is given by  $1 - v\gamma$ , which equals 0.0065. Thus, the *direct* effect on aggregate demand of a 10 percent temporary (one-quarter) blip in the bubble would be an increase of 0.065 percent, a small figure. Adding the indirect effect resulting from the induced increase in current output, the total effect is given by  $(1 - v\gamma)/v\gamma = 0.00663$ , a value only marginally larger. On the other hand, starting from the bubbleless BGP ( $\Lambda\Gamma v = \beta$ ), the implied total effect of a hypothetical aggregate bubble that survives into the following period with probability  $\delta$  is given by the multiplier  $(1 - \beta\gamma)(1 - \beta\gamma\delta)/\beta\gamma(1 - \delta)$  which can be made arbitrarily large as  $\delta \rightarrow 1$ . Thus, it is clear that the size of the effects of a change in the aggregate bubble on current aggregate demand and output hinge critically on its perceived persistence and, thus, on its impact on future income and, as a result, on current fundamental wealth.

Next I analyze the model's equilibrium under the assumption of fully flexible prices, before turning to the (more interesting) case of sticky price equilibria.

## 4.1 Equilibrium under Flexible Prices

Note that under *flexible prices*  $\hat{y}_t = 0$  for all  $t$ , i.e. the output gap is zero at all times. Given (29), that outcome requires that any changes in the aggregate bubble are exactly offset by changes in fundamental wealth of the opposite sign, i.e.  $\hat{x}_t = -\hat{q}_t^B$  for all  $t$ , in order to stabilize aggregate demand. This is achieved through an appropriate adjustment of the real interest rate.

Under what conditions can (non-explosive) aggregate bubble fluctuations arise in a neighborhood of a given BGP? Assuming  $E_t\{\hat{u}_{t+1}\} = 0$  for all  $t$ , equilibrium conditions (31) and (32) can be combined to derive the equilibrium process for the aggregate bubble:

$$\hat{q}_t^B = \chi E_t\{\hat{q}_{t+1}^B\}$$

where  $\chi \equiv \left(\frac{\Lambda\Gamma v}{\beta}\right) \frac{\Lambda\Gamma(1-\beta\gamma)+\gamma(\beta-\Lambda\Gamma v)}{1-\Lambda\Gamma v\gamma} > 0$  is a function of the steady state real interest rate. If  $\chi < 1$  the only non-explosive solution to the previous difference equation is given by  $\hat{q}_t^B = 0$  for all  $t$ , which implies the absence of bubble fluctuations. On the other hand, if  $\chi > 1$  there exist equilibria characterized by stationary bubble fluctuations in a neighborhood of a bubbly BGP. It can be easily checked that  $\chi$  is decreasing in  $r$  over the range of that variable consistent with the existence of a

bubbly BGP. In addition,  $\lim_{r \rightarrow \Gamma v / \beta - 1} \chi = \beta / v > 1$  and  $\lim_{r \rightarrow g} \chi = (v / \beta)(1 - v) / (1 - v\gamma) \in (0, 1)$ . By continuity, there exists a threshold interest rate  $\bar{r} \in (\Gamma v / \beta - 1, g)$  with associated bubble size  $\bar{q}^B$  such that  $\chi > 1$  for any  $\bar{r} \in (\Gamma v / \beta - 1, \bar{r})$  (with corresponding  $q^B \in (0, \bar{q}^B)$ ). This determines a continuum of bubbly BGPs in a neighborhood of which stationary bubble-driven equilibria exist, with bubble fluctuations described by the difference equation:

$$\widehat{q}_t^B = (1/\chi)\widehat{q}_{t-1}^B + \xi_t$$

where  $\chi > 1$  and  $\xi_t \equiv \widehat{u}_t + b_t - E_{t-1}\{b_t\}$  is the innovation in the aggregate bubble, unrelated to fundamentals. Along that equilibrium the real interest rate is given by:

$$\widehat{r}_t = \Theta \widehat{q}_t^B \quad (35)$$

where  $\Theta \equiv \frac{(1 - \Lambda \Gamma v \gamma)(1 - \frac{\Lambda \Gamma v \gamma}{\chi})}{\Lambda \Gamma v \gamma} > 0$ , i.e. the real rate comoves positively with the bubble, as required in order to fully stabilize aggregate demand and output.

Note that both the bubble and the real interest rate are independent of monetary policy, since the latter is neutral under flexible prices. The implied equilibrium path of the real interest rate, combined with the assumed interest rate rule, can be used to determine equilibrium inflation, which must solve the difference equation:

$$\phi_\pi \pi_t = E_t\{\pi_{t+1}\} + (\Theta - \phi_q)\widehat{q}_t^B$$

Assuming  $\phi_\pi > 1$ , the previous equation has a unique non-explosive solution, given by:

$$\pi_t = \frac{\chi(\Theta - \phi_q)}{\chi\phi_\pi - 1} \widehat{q}_t^B$$

Thus, an rise in the aggregate bubble may increase or decrease inflation depending on the size of  $\phi_q$  relative to  $\Theta$ . In particular, if the central bank does not respond to the bubble ( $\phi_q = 0$ ) inflation will comove positively with the bubble.

The fact that, under flexible prices, aggregate bubble fluctuations (i) have no effect on output and (ii) are independent of monetary policy, limits the interest of that environment. By contrast, and as discussed below, bubble fluctuations, when interacting with nominal rigidities, generate fluctuations in aggregate demand and output, transmitted through the implied wealth effects on consumption. Furthermore, the presence of sticky prices makes room for monetary policy to play a role in the stabilization of such bubble-driven output fluctuations. Next I turn to the analysis and characterization of equilibria under sticky prices.

## 4.2 Equilibrium under Sticky Prices

Next I turn to the analysis of the properties of equilibria in a neighborhood of a BGP in the presence of sticky prices. The latter make it possible for aggregate demand changes resulting from bubble fluctuations to have effects on output. I analyze separately the case of fluctuations around bubbleless and bubbly BGPs.

## 4.3 Equilibrium Dynamics around a Bubbleless BGP

As shown above,  $\Lambda\Gamma v = \beta$  along a bubbleless BGP (i.e.  $q^B = 0$ ). Imposing those conditions on the system of difference equations describing the non-policy block of the economy yields:

$$\hat{y}_t = (1 - \beta\gamma)(q_t^B + \hat{x}_t) \quad (36)$$

$$\hat{x}_t = \beta\gamma E_t\{\hat{x}_{t+1}\} + \hat{y}_t - \frac{\beta\gamma}{1 - \beta\gamma}(\hat{i}_t - E_t\{\pi_{t+1}\}) \quad (37)$$

$$\pi_t = (\beta\gamma\theta_I/\theta) E_t\{\pi_{t+1}\} + \kappa\hat{y}_t \quad (38)$$

$$q_t^B = (\beta/v)E_t\{b_{t+1}\} \quad (39)$$

$$q_t^B = b_t + u_t \quad (40)$$

where  $(q_t^B, u_t) \geq 0$  for all  $t$ .

Next I analyze separately bubbly and bubbleless equilibria, both in a neighborhood of the bubbleless BGP, and discuss the role of monetary policy.

### 4.3.1 Bubbleless Equilibria

I start by analyzing the case of *bubbleless equilibria*, i.e.  $q_t^B = 0$  for all  $t$ . The non-policy block of the model now simplifies to:

$$\hat{y}_t = E_t\{\hat{y}_{t+1}\} - (\hat{i}_t - E_t\{\pi_{t+1}\}) \quad (41)$$

$$\pi_t = \Phi E_t\{\pi_{t+1}\} + \kappa\hat{y}_t \quad (42)$$

where  $\Phi \equiv \beta\gamma\theta_I/\theta$ . Note that the previous system has a structure "similar" to that describing the non-policy block of the standard New Keynesian model. As an aside, note that dynamic IS equation (41) is identical to the one in the model with an infinitely-lived representative consumer

and, accordingly, gives rise to the same "forward guidance puzzle" that emerges in that model:<sup>26</sup> the "extra discounting" resulting from agents' shorter horizons in the present model and which is captured in a partial equilibrium condition like (30) vanishes once general equilibrium condition is imposed ( $\widehat{y}_t = \widehat{c}_t$ ).<sup>27</sup>

In order to close the model I assume a simple interest rate rule of the form:

$$\widehat{i}_t = \phi_\pi \pi_t \quad (43)$$

where  $\phi_\pi \geq 0$ . Using (43) to eliminate the interest rate in (41) one can write the resulting system (after some manipulation) as:

$$\begin{bmatrix} \widehat{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} E_t\{\widehat{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} \quad (44)$$

where

$$\mathbf{A} \equiv \Omega \begin{bmatrix} 1 & 1 - \phi_\pi \beta \gamma \theta_I / \theta \\ \kappa & \kappa + \beta \gamma \theta_I / \theta \end{bmatrix}$$

with  $\Omega \equiv 1/(1 + \kappa \phi_\pi)$ .

The solution to (44) is (locally) unique (and equal to  $\widehat{y}_t = \pi_t = 0$  for all  $t$ ) if and only if the two eigenvalues of matrix  $\mathbf{A}$  are inside the unit circle. As shown in the appendix, this will be the case if the following condition is satisfied:

$$\phi_\pi > \max \left[ 1, \frac{\beta \gamma \theta_I / \theta - 1}{\kappa} \right] \quad (45)$$

If the previous condition is satisfied, and in the absence of exogenous shocks, the only (bubbleless) equilibrium that remains in a neighborhood of the steady state is given by  $\widehat{y}_t = \pi_t = 0$  for all  $t$ .

If  $\theta > \frac{\beta \gamma \theta_I}{1 + \kappa}$ , i.e. if average price stickiness (relative to that of incumbent firms) is above the threshold value  $\beta \gamma / (1 + \kappa) \in (0, 1)$ , uniqueness condition (45) simplifies to:

$$\phi_\pi > 1$$

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<sup>26</sup>See, e.g. McKay et al. (2016).

<sup>27</sup>This finding calls into question the solution of the "forward guidance puzzle" proposed by Del Negro et al (2015) based on the assumption of finitely-lived agents.



which corresponds to the usual "Taylor principle": the central bank should adjust the nominal rate "more than one-for-one" in response to changes in inflation in order to guarantee a (locally) unique equilibrium.

On the other hand if  $\theta < \frac{\beta\gamma\theta_I}{1+\kappa}$  then the threshold value for coefficient  $\phi_\pi$  is above one, and the resulting uniqueness condition  $\phi_\pi > (\beta\gamma\theta_I/\theta - 1)/\kappa > 1$  can be thought of as a "reinforced Taylor principle," requiring that the central bank adjust the policy rate more aggressively than implied by the usual "more than one-for-one" condition in order to guarantee a unique equilibrium. Note that a low value for average price stickiness  $\theta$ , combined with large values for  $\beta$ ,  $\gamma$  and  $\theta_I$  makes current inflation more sensitive to expectations (i.e. "more forward-looking"), and hence a stable backward solution to (44) becomes more likely, especially when  $\kappa$  is small, unless the central bank responds more aggressively to inflation thus inducing a sufficiently large decrease in the output gap.<sup>28</sup>

If condition (45) is not satisfied, there exist equilibria involving stationary sunspot fluctuations around the bubbleless BGP, driven by self-fulfilling revisions of expectations. If  $\phi_\pi < 1$  then only one eigenvalue of  $\mathbf{A}$  is larger than one, implying indeterminacy of dimension one.<sup>29</sup> On the other hand, if  $1 \leq \phi_\pi < (\beta\gamma\theta_I/\theta - 1)/\kappa$  both eigenvalues of  $\mathbf{A}$  have moduli larger than one, with the resulting indeterminacy being two-dimensional, a case that can be ruled out in the New Keynesian model with a representative agent. See the appendix for a discussion of the representation of such equilibria.

The previous results point to a first dimension in which the introduction of finite lives and an OLG structure in an otherwise standard New Keynesian model can potentially affect the properties of its associated equilibrium dynamics and their connection to monetary policy, *even in the absence of bubbles*.

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<sup>28</sup>Note, in particular, that in the limiting case of full price flexibility for newly created firms ( $\theta_N = 0$ ), the uniqueness condition can be written as

$$\phi_\pi > \max \left[ 1, \frac{\beta/v - 1}{\kappa} \right]$$

implying a "reinforced Taylor principle" if  $v > \beta/(1 + \kappa)$ .

<sup>29</sup>Such sunspot equilibria are similar in nature to those analyzed in, e.g. Clarida, Galí and Gertler (2000).

### 4.3.2 Bubbly Equilibria

Next I consider the possibility of bubbly equilibria in a neighborhood of the bubbleless BGP. To illustrate the possibility of bubble-driven fluctuations and the potential role of monetary policy in shaping those fluctuations, I specify a process for the bubble which satisfies the conditions above while displaying a boom-bust pattern characteristic of conventional accounts of historical bubble episodes. Consider a bubbly asset with positive value that is introduced in a fully unanticipated way by a newly born cohort (say, in period 0). Conditional on not bursting the bubble evolves over time according to the equation

$$q_t^B = \frac{v}{\beta\delta} q_{t-1}^B \quad (46)$$

for  $t = 1, 2, 3, \dots$  where  $\delta < v/\beta < 1$  is the bubble's survival probability, assumed to be constant. Thus, with probability  $\delta$  the bubble grows at a gross rate  $v/\beta\delta > 1$ . With probability  $1 - \delta$  the bubble bursts, and does not re-emerge again. Note that the previous specification satisfies (39) and (40) as well as the non-negativity condition. Furthermore, the assumption  $v/\beta < 1$  guarantees that  $\lim_{T \rightarrow \infty} E_t\{q_{t+T}^B\} = 0$ .

Combining (36), (41) and the process for the bubble assumed above, one can derive the dynamic IS equation:

$$\hat{y}_t = E_t\{\hat{y}_{t+1}\} - (\hat{i}_t - E_t\{\pi_{t+1}\}) + \Theta q_t^B \quad (47)$$

where now  $\Theta \equiv (1 - \beta\gamma)(1 - v\gamma)/\beta\gamma > 0$ . Note that one can rewrite (47) as follows:

$$\hat{y}_t = E_t\{\hat{y}_{t+1}\} - (\hat{i}_t - E_t\{\pi_{t+1}\} - \hat{r}_t^n)$$

where  $\hat{r}_t^n = \Theta q_t^B$  has the interpretation of a *natural* rate of interest, i.e. the equilibrium real interest rate that would prevail under flexible prices.<sup>30</sup>

The previous equation, together with (13) and a process for the bubble satisfying (39) and (40) describe the non-policy block of the economy. In addition, I assume an interest rate rule of the form

$$\hat{i}_t = \phi_\pi \pi_t + \phi_q q_t^B \quad (48)$$

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<sup>30</sup>Note that the previous relationship corresponds to that shown in (35) when  $\Theta$  in the latter is evaluated at the bubbleless BGP.

where  $\phi_\pi > 1$ , which guarantees the (local) uniqueness of the equilibrium in the absence of a bubble.

By eliminating the interest rate, we can rewrite the corresponding system of difference equations more compactly as

$$\begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} E_t\{\hat{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix} + \mathbf{B}q_t^B$$

where  $\{q_t^B\}$  evolves according to (46),  $\mathbf{A}$  and  $\Omega$  are defined as above, and  $\mathbf{B} \equiv \Omega(\Theta - \phi_q)[1 \quad \kappa]'$ .

Based on the analysis above, the condition for (local) uniqueness of the equilibrium, conditional on a given bubble process, is given by (45) and is thus independent of the presence of the bubble or the policy response to it, as measured by  $\phi_q$ . The latter parameter, however, is key in determining the effects of the bubble on the real economy.

Note that in the case considered here, and up to a first order approximation, there is no feedback from monetary policy to the bubble (since  $q^B = 0$ ). Hence, fluctuations in the bubble can be viewed as playing a role similar to an exogenous demand shock, albeit one that must satisfy conditions (39) and (40), as well as the non-negativity condition.

Assuming that the condition for a unique equilibrium (45) is satisfied, one can use the method of undetermined coefficients to derive closed-form expressions for inflation and the output gap as a function of the bubble:

$$\begin{aligned} \hat{y}_t &= (1 - v\gamma\theta_I/\theta)\Psi(\Theta - \phi_q)q_t^B \\ \pi_t &= \kappa\Psi(\Theta - \phi_q)q_t^B \end{aligned}$$

where  $\Psi \equiv 1/[(1 - v\gamma\theta_I/\theta)(1 - v/\beta) + \kappa(\phi_\pi - v/\beta)] > 0$ . Thus, the emergence of a bubble will trigger movements in inflation and the output, as long as  $\phi_q \neq \Theta$ . Furthermore, note that  $v\gamma\theta_I/\theta \leq 1$ , implying that comovement between output and inflation is always positive. On the other hand the sign of the response of both output and inflation to fluctuations in the bubble is given by the sign of  $\Theta - \phi_q$ . Thus, if  $\phi_q < \Theta$ , both variables will increase with the bubble.<sup>31</sup> On the other hand, if the central bank "overreacts" to the bubble ( $\phi_q > \Theta$ ), both inflation and the output gap will decline as the bubble grows. By setting  $\phi_q = \Theta$  the central bank will succeed in insulating the output gap and inflation from fluctuations in the bubble. Note that such a policy

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<sup>31</sup>Strictly speaking this is true as long as  $\theta_N > 0$ . If  $\theta_N = 0$  (i.e. if there is no backwardlooking behavior on the part of new firms) then  $\theta = v\gamma\theta_I$  and output will be invariant to fluctuations in the bubble.

is equivalent to following the rule

$$\widehat{i}_t = \widehat{r}_t^n + \phi_\pi \pi_t$$

where  $\widehat{r}_t^n = \Theta q_t^B$ .

Such a policy can be thought of as a "leaning against the bubble" rule, since it requires raising interest rates when the bubble is growing, and lowering them when it bursts. That policy, however, *does not have any effect on the bubble itself*, it just offsets the effects of the bubble on aggregate demand. In fact, it should be clear that the same outcome can be achieved by *any other policy rule* that succeeds in stabilizing inflation, for in that case, and independently of how the policy is implemented,  $\widehat{i}_t = \Theta q_t^B$  will have to hold in equilibrium. An example of such a policy is given by the limit of the simple rule  $\widehat{i}_t = \phi_\pi \pi_t$  as  $\phi_\pi \rightarrow +\infty$ . It follows that observability of the bubble and a systematic response to the latter is not a requirement for insulating the economy from bubble fluctuations, at least in the environment analyzed here. That result is reminiscent of the main findings in Bernanke and Gertler (1999, 2001). On the other hand, an inappropriate choice of coefficient  $\phi_q$  and/or a "more flexible" form of inflation targeting will fail to insulate the "real economy" from the booms and busts experienced by the bubble.

In the previous example, a single boom and bust episode is triggered by a one-off bubble. For an example with *recurrent* bubble-driven booms and eventual collapses consider an economy where the bubble evolves exogenously according to the following process.

$$q_t^B = \begin{cases} \frac{v}{\beta\delta} q_{t-1}^B + u_t & \text{with probability } \delta \\ u_t & \text{with probability } 1 - \delta \end{cases} \quad (49)$$

where  $\{u_t\}$  follows a white noise process with positive support and constant mean  $\bar{u} \gtrsim 0$ . Again, the previous process satisfies (39) and (40), as well as the non-negativity condition. Furthermore,  $E_t\{q_{t+1}^B\} = (v/\beta)q_t^B + \bar{u}$ , and hence  $\lim_{T \rightarrow \infty} E_t\{q_{t+T}^B\} = \bar{u}/(1 - v/\beta)$ , the size of the bubble converges in expectation to a value arbitrarily close to zero. Equilibrium output gap and inflation can be shown to be given by

$$\widehat{y}_t = \psi_y + (1 - v\gamma\theta_I/\theta)\Psi(\Theta - \phi_q)q_t^B$$

$$\pi_t = \psi_\pi + \kappa\Psi(\Theta - \phi_q)q_t^B$$

where  $\lim_{\bar{u} \rightarrow 0} \psi_y = \lim_{\bar{u} \rightarrow 0} \psi_\pi = 0$ . Note that all the qualitative findings regarding the role of

monetary policy in shaping the effects of bubble fluctuations found in the discussion above apply also to the present case.

Figure 2 displays a simulated equilibrium path generated by the previous model. In addition to the baseline parameter settings introduced above, I set  $\delta = 0.95$  as values describing the process for the bubble and draw  $\{u_t\}$  from an exponential distribution with mean 0.001. Furthermore, I set  $\kappa = 0.0424$  as a baseline value for the slope of the New Keynesian Phillips curve. That value is consistent with  $\theta_I = \theta = 0.75$  and  $\varphi = 0.5$ . The former setting is consistent with an average duration of individual prices of 4 quarters, in accordance with much of the micro evidence. Note that, lacking evidence on the extent of "backward-looking behavior" among new firms, I assume  $\theta_N = \theta_I = \theta$ . The setting for  $\varphi$  is consistent with (6) and the observation that the standard deviation of the (log) real wage is roughly half the size that of (log) hours worked (both HP-detrended), based on postwar U.S. data. Finally, I assume a "flexible inflation targeting" policy, with  $\phi_\pi = 1.5$  and  $\phi_q = 0$ . The simulated fluctuations display a characteristic "boom-bust" pattern for the three variables. As noted above, though, the size of the fluctuations in output and inflation is negligible, despite the substantial size of the bubble fluctuations.

Figure 3 shows the standard deviation of output as a function of  $\phi_\pi$  and  $\phi_q$ , given a standard deviation of the bubble normalized to one percent. The figure illustrates graphically the dangers of "leaning against the bubble" monetary policies discussed above: the "wrong" choice of coefficient  $\phi_q$  will easily lead to an increase in output volatility, which contrasts with the unambiguous stability gains from a stronger anti-inflationary stance.

#### 4.4 Equilibrium Dynamics around a Bubbly BGP

Next I analyze the properties of the model's equilibrium dynamics in a neighborhood of a *bubbly* BGP ( $q^B > 0$ ). For simplicity I assume that the newly created bubbles follow a martingale difference process about a constant mean  $\bar{u}$ , so that  $E_t\{\widehat{u}_{t+1}\} = 0$  for all  $t$ . By combining (29), (31), (32), (33) and (28) the following version of the dynamic IS equation can be derived:

$$\widehat{y}_t = \frac{\Lambda\Gamma v}{\beta} E_t\{\widehat{y}_{t+1}\} - \frac{\Upsilon v}{\beta} (\widehat{i}_t - E_t\{\pi_{t+1}\}) + \Theta \widehat{q}_t^B \quad (50)$$

where  $\Theta \equiv \frac{(1-\beta\gamma)(1-v\gamma)}{\beta\gamma} > 0$  and  $\Upsilon \equiv 1 + \frac{(1-\beta\gamma)(\Lambda\Gamma-1)}{1-\Lambda\Gamma v\gamma} \geq 1$ . The previous equation has a form similar to the dynamic IS equation in the standard New Keynesian model with preference shocks, with

fluctuations in the bubble playing the role of the latter. In contrast with the "standard" model though, note that the coefficients on expected inflation and interest rate will be different from unity. In particular, the fact that  $\Lambda\Gamma v/\beta < 1$  in a bubbly BGP makes the influence of an expected change in the interest rate on current output decline with the horizon of the policy intervention, thus overcoming the extreme version of the so-called "forward guidance puzzle."<sup>32</sup> Note also that the largest possible theoretical value for coefficient  $\Lambda\Gamma v/\beta$  determining the rate of decline is  $v$ .<sup>33</sup> Under the baseline calibration for that parameter that implies a discount factor of 0.9 applied to the interest rate changes anticipated to take place in 10 years time, thus pointing a substantial effectiveness of forward guidance policies, despite the discounting.

In addition to (50) the non-policy block of the system describing the equilibrium dynamics around a bubbly BGP is given by the bubble equation (32) and the New Keynesian Phillips curve (13), which are reproduced here for convenience:

$$\begin{aligned}\widehat{q}_t^B &= \Lambda\Gamma E_t\{\widehat{q}_{t+1}^B\} - q^B(\widehat{i}_t - E_t\{\pi_{t+1}\}) \\ \pi_t &= \Phi E_t\{\pi_{t+1}\} + \kappa\widehat{y}_t\end{aligned}$$

where  $\Phi \equiv \Lambda\Gamma v\gamma\theta_I/\theta$  and  $\kappa \equiv \varphi(1-\theta)(1-\Lambda\Gamma v\gamma\theta_I)/\theta$ .

Note that, in contrast with the system describing the equilibrium around the bubbleless steady state, the size of the bubble has now an *endogenous* component, which depends on the real interest rate.

Again, I assume the central bank follows the interest rate rule (48), repeated here for convenience:

$$\widehat{i}_t = \phi_\pi\pi_t + \phi_q q_t^B$$

Note that by setting  $\phi_q = \frac{\Theta\beta}{\Upsilon v}$  the central bank can fully offset the effects of the bubble on aggregate demand, output and inflation. In that case, the equilibrium dynamics are described by

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<sup>32</sup>The previous result can be shown to be linked to the fact that the factor  $\Lambda\Gamma v\gamma$  at which future income is discounted when determining fundamental wealth (see (31)) differs from the term  $\beta\gamma$  in the marginal propensity to consume in the presence of the bubble in the BGP, as implied by (24). This will also be the case in the presence of an asset other than the bubble which is in (i) positive net supply, (ii) it is not a claim to future output, and (iii) which has a positive value for current consumers (net of liabilities associated with it, e.g. taxes). An example of such an asset is government debt in the presence of finite horizons, i.e. when Ricardian equivalence does not hold. The presence of such debt is key in accounting for the muting of the "forward guidance puzzle" in Del Negro et al. (2015).

<sup>33</sup>This would correspond to the BGP with the highest admissible interest rate ( $\Lambda\Gamma = 1$ ) and  $\beta = 1$ .

the system

$$\begin{bmatrix} \widehat{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A}_* \begin{bmatrix} E_t\{\widehat{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

where

$$\mathbf{A}_* \equiv \Omega_* \begin{bmatrix} \frac{\Lambda\Gamma v}{\beta} & \frac{\Upsilon v}{\beta}(1 - \Phi\phi_\pi) \\ \frac{\Lambda\Gamma v}{\beta}\kappa & \frac{\Upsilon v}{\beta}\kappa + \Phi \end{bmatrix}$$

with  $\Omega_* \equiv \frac{1}{1+(\Upsilon v/\beta)\kappa\phi_\pi}$ . The necessary and sufficient condition for (local) uniqueness can be shown to be given by:

$$\phi_\pi > 1 - \frac{(1 - \Phi)(\beta - \Lambda\Gamma v)}{\kappa\Upsilon v} \quad (51)$$

The threshold value for  $\phi_\pi$  defining the determinacy region, given by the right hand side of the previous expression is exactly equal to unity at the bubbleless BGP (where  $\Lambda\Gamma v/\beta = 1$ ), in a way consistent with the findings above. At the other extreme, corresponding to a BGP with the largest possible bubble and interest rate (implying  $\Lambda\Gamma = 1$ ) that lower bound is given by  $1 - \frac{(1-v\gamma\theta_I/\theta)(\beta-v)}{\kappa v}$  which is likely to be very close to one under any reasonable calibration (under the baseline calibration is approximately equal to 0.999).

Thus, an interest rate rule  $\widehat{i}_t = \phi_\pi\pi_t + \frac{\Theta\beta}{\Upsilon v}q_t^B$  with  $\phi_\pi$  satisfying (51) guarantees full stabilization of the output gap and inflation in the face of fluctuations in the aggregate bubble. Under that rule, the aggregate bubble will evolve according to the difference equation:

$$\widehat{q}_t^B = (1/\chi)\widehat{q}_{t-1}^B + \xi_t \quad (52)$$

where  $\chi \equiv \left(\frac{\Lambda\Gamma v}{\beta}\right) \frac{\Lambda\Gamma(1-\beta\gamma)+\gamma(\beta-\Lambda\Gamma v)}{1-\Lambda\Gamma v\gamma} > 0$  and  $\xi_t \equiv \widehat{b}_t - E_{t-1}\{\widehat{b}_t\} + \widehat{u}_t$  is the aggregate bubble innovation. Note that the previous process for the bubble corresponds to that obtained above for the flexible price equilibrium. As in that case,  $\chi > 1$  must be assumed in order to guarantee that the bubble doesn't drift away permanently from its value along the BGP, which in turn requires that the latter is associated with not too high a real interest rate and bubble size (formally,  $r < \bar{r}$  and  $q^B < \bar{q}^B$ ; see discussion of flexible price equilibrium above). As long as that condition is satisfied, the aggregate bubble may display non-explosive fluctuations around its BGP value as described by (52).

Note however that, as in the case of fluctuations around the bubbleless BGP analyzed above, an identical outcome will obtain if the central bank follows any other rule that succeeds in stabilizing inflation (and hence the output gap), for in that case (50) requires that in equilibrium

$\hat{i}_t - E_t\{\pi_{t+1}\} = \hat{i}_t = \frac{\Theta\beta}{\Upsilon v} q_t^B$ . Thus, as in the bubbleless BGP case, we see that in the face of bubble fluctuations a "leaning against the bubble" policy is not required to insulate output and inflation from those fluctuations and, in any case, that policy doesn't work through by dampening fluctuations in the bubble.

What are the implications of more general calibrations of the interest rate rule (48)? Using (??) to eliminate the interest rate in (50) and (32), we can rewrite the system as follows:

$$\mathbf{A}_0 \mathbf{x}_t = \mathbf{A}_1 E_t\{\mathbf{x}_{t+1}\}$$

where  $\mathbf{x}_t \equiv [\hat{y}_t, \pi_t, \hat{q}_t^B]'$  and

$$\mathbf{A}_0 \equiv \begin{bmatrix} 1 & \frac{\Upsilon v}{\beta} \phi_\pi & -\Theta + \frac{\Upsilon v}{\beta} \phi_q \\ -\kappa & 1 & 0 \\ 0 & q^B \phi_\pi & 1 + q^B \phi_q \end{bmatrix} ; \quad \mathbf{A}_1 = \begin{bmatrix} \frac{\Lambda \Gamma v}{\beta} & \frac{\Upsilon v}{\beta} & 0 \\ 0 & \Phi & 0 \\ 0 & q^B & \Lambda \Gamma \end{bmatrix}$$

The solution to the previous dynamical system is (locally) unique if and only if the three eigenvalues of the companion matrix  $\mathbf{A} \equiv \mathbf{A}_0^{-1} \mathbf{A}_1$  lie inside the unit circle. If that condition is satisfied, the equilibrium is given by  $\hat{y}_t = 0$ ,  $\pi_t = 0$  and  $\hat{q}_t^B = 0$  for all  $t$ . That uniqueness condition, however, is far from being guaranteed in all cases even for empirically plausible values of the rule coefficients. This is illustrated in Figure 4, which displays the uniqueness and indeterminacy regions on the  $(\phi_\pi, \phi_q)$  plane, under alternative assumptions regarding the real interest rate (and, hence, the bubble size) in the underlying BGP, given the baseline calibration introduced above. Note that when the BGP is close to the bubbleless one (with  $r = 0.00335$ , a value near its lower bound consistent with a BGP), indeterminacy is pervasive, independently of the value of the rule coefficients. Furthermore, if  $\phi_\pi < 1$ , that indeterminacy is two dimensional, a property not found in the standard model under the present rule. As we consider BGPs with larger bubble-output ratios and higher real interest rates, we see that a region with a unique equilibrium emerges, and grows larger as  $r$  increases. Under the largest interest rate consistent with a bubbly BGP ( $r = g = 0.004$ ), the equilibrium is seen to be determinate even for values of  $\phi_\pi$  less than one, as long as  $\phi_q$  is positive.

As an illustration, Figure 5 displays simulated stationary fluctuations corresponding to a model calibration consistent with one-dimensional indeterminacy. In particular I assume  $r = 0.0035$  (the estimated average interest rate in postwar U.S.),  $\phi_\pi = 1.5$  (the value proposed in Taylor (1993) and often used in related exercises) and  $\phi_q = 0$  (a likely relevant value for the Fed and other central



banks, at least before the financial crisis). Note that in the standard New Keynesian model, the Taylor principle condition  $\phi_\pi > 1$  would guarantee a unique equilibrium. By contrast, in the present model such a rule is seen to be consistent with fluctuations that are highly persistent and involve "proportional" variations in output, inflation and the bubble.

Figure 6a displays the standard deviation of the output gap as a function of  $\phi_\pi$  and  $\phi_q$ , given  $r = 0.0035$ , and keeping the remaining parameter settings unchanged. The qualitative picture is similar to that obtained for the case of fluctuations around the bubbleless BGP, pointing to the potential destabilizing effects of "leaning against the bubble" policies that are not surgically calibrated, and which stand in contrast to the more robust gains of inflation targeting policies. Figure 6b displays the standard deviation of the bubble, as a function of  $\phi_\pi$  and  $\phi_q$ , and under an identical calibration of the remaining parameters. The figure illustrates a hidden peril of "leaning against the bubble" policies: contrary to conventional wisdom, such policies may end up increasing the size of bubble fluctuations. That outcome, already uncovered in Galí (2014) in the context of a two-period monetary OLG model, follows from the requirement that any existing bubble grows (in expectation) at the rate of interest. Thus, if the latter increases with the bubble, the size of bubble fluctuations may be amplified.

## 5 Concluding Comments

The New Keynesian model remains the workhorse framework for monetary policy analysis, even though it is unsuitable –in its standard formulation– to accommodate the existence of asset price bubbles and hence to address one of the key questions facing policy makers, namely, how monetary policy should respond to those bubbles. That shortcoming, however, is not tied to any key ingredient of the model (e.g. staggered price setting), but to the convenient (albeit unrealistic) assumption of an infinite-lived representative consumer. In the present paper I have developed an extension of the basic New Keynesian model featuring overlapping generations, finite lives and (stochastic) transitions to inactivity. In contrast with the standard model, the proposed framework allows for the existence of rational expectations equilibria with asset price bubbles. In particular, plausible calibrations of the model's parameters are consistent with the existence of a continuum of bubbly balanced growth paths, as well as a bubbleless one (which always exists). When combined with sticky prices, fluctuations in the size of the aggregate bubble, possibly unrelated to changes

in fundamentals, are a potential source of fluctuations in output and inflation.

The analysis of the properties of the model has yielded several insights regarding the implications for monetary policy, which I briefly summarize next.

Firstly, the introduction of an overlapping generations structure, even when one abstracts from the possibility of bubbly equilibria, may change the conditions under which simple interest rate rules will guarantee equilibrium uniqueness. In particular, under some parameter configurations, the so-called Taylor principle has to be "reinforced," calling for a stronger anti-inflationary stance in order to avoid equilibrium indeterminacy.

Secondly, a "leaning against the bubble" interest rate policy, if precisely calibrated, may succeed in insulating output and inflation from aggregate bubble fluctuations. However, mismeasurement of the bubble or an inaccurate "calibration" of the policy response to its fluctuations may end up destabilizing output and inflation. On the other hand, and contrary to conventional wisdom, a "leaning against the bubble" policy, even when precisely implemented, does not guarantee the elimination of the bubble or the dampening of its fluctuations. In fact, under some conditions such a policy may end up *increasing* the volatility and persistence of bubble fluctuations. By way of contrast, a policy that targets inflation directly may attain the same stabilization objectives without the risks associated with a "direct" response to the bubble.

Four additional remarks, pointing to possible future research avenues are in order. Firstly, the analysis of the equilibrium dynamics above has assumed "rationality" of asset price bubbles. That assumption underlies the equilibrium conditions that individual and aggregate bubbles must satisfy, i.e. (17) and (19), respectively, and the implied log-linear approximation (32). However, the remaining equilibrium conditions, including the modified dynamic IS equations, *are still valid* even if the process followed by the aggregate bubble is inconsistent with a *rational* bubble. That observation opens the door to analyses of the macroeconomic effects and policy implications of *alternative* specifications of the aggregate bubble's behavior.

Secondly, the analysis above suggests that, in the absence of other frictions, the rise and fall of an aggregate bubble is likely to have small quantitative effects on aggregate demand and, hence, on output and inflation. To the extent that the historical evidence points to a larger macro impact of bubbles, there may be additional channels through which that impact operates. The work of Farhi and Tirole (2011), Martin and Ventura (2012), and Asriyan et al. (2016), emphasizing the

interaction of the bubble with financial frictions, points to a possible source of larger quantitative effects that one might be able to incorporate in the above framework.

Thirdly, balanced growth paths characterized by a larger bubble size are associated with a higher real interest rate. Thus, the presence of a bubble along a BGP should make it less likely for the zero lower bound on the nominal interest rate to become binding, *ceteris paribus* and conditional on the bubble not bursting. Similarly, the bursting of the bubble would bring along a reduction in the natural rate of interest that could pull the interest rate toward the zero lower bound. The analysis of the interaction of bubble dynamics with the zero lower bound seems an avenue worth exploring in future research.

Finally, the analyses of the equilibrium dynamics above has assumed that the central bank takes as given the BGP on which the economy settles and, hence, its associated real interest rate. That assumption is reflected in the implicit exogeneity of the real interest rate term embedded in the policy rule through the term  $\hat{i}_t \simeq i_t - r$ . But while that assumption is a natural one in the context of economies whose real interest rate along a BGP is uniquely pinned down (as it is the case in the standard New Keynesian model with a representative consumer), it is no longer so in an economy like the one analyzed above, in which a multiplicity of real interest rates may be consistent with a perfect foresight BGP. Future research should explore the implications of relaxing the assumption of a policy invariant steady state real interest rate.

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## TECHNICAL APPENDIX

### 1. Transversality condition in a Bubbly BGP

The consumption function for an active individual born in period  $s$  is:

$$C_{t+T|s}^a = (1 - \beta\gamma) \left[ A_{t+T|s}^a + \frac{W_{t+T}N/\alpha}{1 - \Lambda\Gamma v\gamma} \right]$$

In particular:

$$C_{t|s}^a = (1 - \beta\gamma) \left[ A_{t|s}^a + \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} \right]$$

In addition,  $C_{t+T|s} = (\beta/\Lambda)^T C_{t|s}$  thus implying

$$A_{t+T|s}^a = (\beta/\Lambda)^T \left[ A_{t|s}^a + (1 - (\Lambda\Gamma/\beta)^T) \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} \right]$$

On the other hand, for a retired individual

$$C_{t|s}^r = (1 - \beta\gamma) A_{t|s}^r$$

Using the fact that  $C_{t|s}^r = C_{t|s}^a$  we have:

$$A_{t|s}^r = A_{t|s}^a + \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma}$$

The transversality condition for an active individual takes the form:

$$\begin{aligned} \lim_{T \rightarrow \infty} (\gamma\Lambda)^T E_t \{ A_{t+T|s} \} &= \lim_{T \rightarrow \infty} (\gamma\Lambda)^T [v^T A_{t+T|s}^a + (1 - v^T) A_{t+T|s}^r] \\ &= \lim_{T \rightarrow \infty} (\gamma\Lambda)^T \left[ A_{t+T|s}^a + (1 - v^T) \Gamma^T \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} \right] \\ &= \lim_{T \rightarrow \infty} (\beta\gamma)^T \left[ A_{t|s}^a + \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} (1 - (\Lambda\Gamma/\beta)^T) \right] + \lim_{T \rightarrow \infty} (\Lambda\Gamma\gamma)^T (1 - v^T) \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} \\ &= \frac{W_t N/\alpha}{1 - \Lambda\Gamma v\gamma} \lim_{T \rightarrow \infty} [(\beta\gamma)^T - (\Lambda\Gamma\gamma)^T + (\Lambda\Gamma\gamma)^T - (\Lambda\Gamma v\gamma)^T] \\ &= 0 \end{aligned}$$

where the maintained assumption  $\Lambda\Gamma v\gamma < 1$  has been used.

### 2. Log-linearized individual intertemporal budget constraint

The intertemporal budget constraint for an individual born in period  $s$  and still active in period  $t \geq s$  can be derived by iterating (3) forward to yield:

$$\sum_{k=0}^{\infty} \gamma^k E_t \{ \Lambda_{t,t+k} C_{t+k|s} \} = A_{t|s}^a + \frac{1}{\alpha} \sum_{k=0}^{\infty} (\gamma v)^k E_t \{ \Lambda_{t,t+k} W_{t+k} N_{t+k} \} \quad (53)$$

For retired individuals, the corresponding constraint is:

$$\sum_{k=0}^{\infty} \gamma^k E_t \{ \Lambda_{t,t+k} C_{t+k|s} \} = A_{t|s}^r \quad (54)$$

Letting lowercase letters with a " $\hat{\cdot}$ " symbol denote the log deviation of a variable from its value along a perfect foresight balanced growth path (BGP), the left hand side term of (53) and (54) can be approximated in a neighborhood of the BGP as:

$$\begin{aligned} \sum_{k=0}^{\infty} \gamma^k E_t \{ \Lambda_{t,t+k} C_{t+k|s} \} &\simeq \frac{\Gamma^t \mathcal{C}_{t-s}}{1 - \beta\gamma} + \Gamma^t \sum_{k=0}^{\infty} (\Lambda \Gamma \gamma)^k \mathcal{C}_{t+k-s} E_t \{ \hat{c}_{t+k|s} + \hat{\lambda}_{t,t+k} \} \\ &= \frac{\Gamma^t \mathcal{C}_{t-s}}{1 - \beta\gamma} + \frac{\Gamma^t \mathcal{C}_{t-s}}{1 - \beta\gamma} \hat{c}_{t|s} \end{aligned}$$

where  $\mathcal{C}_j$  denotes consumption of an individual aged  $j$  (normalized by current productivity) along a BGP,  $\hat{\lambda}_{t,t+k} = \log(\Lambda_{t,t+k}/\Lambda^k)$ , and where I have made use of the fact that  $\mathcal{C}_{t+k-s} = [\beta(1+r)/\Gamma]^k \mathcal{C}_{t-s}$  and  $E_t \{ \hat{\lambda}_{t,t+k} + \hat{c}_{t+k|s} \} = \hat{c}_{t|s}$  (resulting from (4)).

The second term on the right hand side of (53), which is relevant only for active individuals, can be approximated around the BGP as:

$$\frac{1}{\alpha} \sum_{k=0}^{\infty} (v\gamma)^k E_t \{ \Lambda_{t,t+k} W_{t+k} N_{t+k} \} \simeq \frac{\Gamma^t \mathcal{W}N}{\alpha(1 - \Lambda \Gamma v \gamma)} + \left( \frac{\Gamma^t \mathcal{W}N}{\alpha} \right) \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \hat{w}_{t+k} + \hat{n}_{t+k} + \hat{\lambda}_{t,t+k} \}$$

where  $\mathcal{W}$  is the wage along the BGP, normalized by productivity.

### 3. Aggregation and derivation of log-linearized aggregate consumption Euler equation

Let  $\mathcal{C}$  denote aggregate consumption (normalized by current productivity), along the BGP. One can derive the approximate relations  $\mathcal{C} \hat{c}_t = (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} \mathcal{C}_{t-s} \hat{c}_{t|s}$  and the BGP relation  $\mathcal{C} = (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} \mathcal{C}_{t-s}$ . Those relations can be used to aggregate the log-linearized individual consumption functions across all individuals to yield:

$$\frac{\Gamma^t \mathcal{C}}{1 - \beta\gamma} + \frac{\Gamma^t \mathcal{C}}{1 - \beta\gamma} \hat{c}_t = (Q_t^F + Q_t^B) + \frac{\Gamma^t \mathcal{W}N}{1 - \Lambda \Gamma v \gamma} + \Gamma^t \mathcal{W}N \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \hat{w}_{t+k} + \hat{n}_{t+k} + \hat{\lambda}_{t,t+k} \}$$

Note also that in a neighborhood of the BGP,

$$\begin{aligned} Q_t^F &= \sum_{k=0}^{\infty} (v\gamma)^k E_t \{ \Lambda_{t,t+k} D_{t+k} \} \\ &\simeq \frac{\Gamma^t \mathcal{D}}{1 - \Lambda \Gamma v \gamma} + \Gamma^t \mathcal{D} \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{d}_{t+k} + \widehat{\lambda}_{t,t+k} \} \end{aligned}$$

where  $\mathcal{D}$  denotes aggregate dividends (normalized by productivity), along a BGP.

Thus, using the approximation  $\widehat{y}_t = (\mathcal{W}N/\mathcal{Y})(\widehat{w}_t + \widehat{n}_t) + (\mathcal{D}/\mathcal{Y})\widehat{d}_t$ , the BGP relation  $\frac{\Gamma^t \mathcal{C}}{1 - \beta \gamma} = Q^B + \frac{\Gamma^t (\mathcal{W}N + \mathcal{D})}{1 - \Lambda \Gamma v \gamma}$ , and the goods market clearing condition  $\mathcal{Y} = \mathcal{C}$ , we obtain the log-linearized aggregate consumption function

$$\begin{aligned} \widehat{c}_t &= (1 - \beta \gamma) \left[ \widehat{q}_t^B + \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{y}_{t+k} + \widehat{\lambda}_{t,t+k} \} \right] \\ &= (1 - \beta \gamma) \left[ \widehat{q}_t^B + \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{y}_{t+k} \} - \sum_{k=1}^{\infty} (\Lambda \Gamma v \gamma)^k \sum_{j=0}^{k-1} E_t \{ \widehat{r}_{t+j} \} \right] \\ &= (1 - \beta \gamma) \left[ \widehat{q}_t^B + \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{y}_{t+k} \} - \frac{\Lambda \Gamma v \gamma}{1 - \Lambda \Gamma v \gamma} \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{r}_{t+k} \} \right] \end{aligned}$$

where  $\widehat{q}_t^B \equiv q_t^B - q^B$ , with  $q_t^B \equiv Q_t^B / (\Gamma^t \mathcal{Y})$  and  $q^B$  its corresponding value along a BGP, and  $\widehat{r}_t \equiv \widehat{i}_t - E_t \{ \pi_{t+1} \}$ .

Note that we can write the consumption function more compactly as:

$$\widehat{c}_t = (1 - \beta \gamma) (\widehat{q}_t^B + \widehat{x}_t) \tag{55}$$

where  $\widehat{x}_t \equiv \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{y}_{t+k} \} - \frac{\Lambda \Gamma v \gamma}{1 - \Lambda \Gamma v \gamma} \sum_{k=0}^{\infty} (\Lambda \Gamma v \gamma)^k E_t \{ \widehat{r}_{t+k} \}$  the non-bubbly component of wealth, expressed in log deviations from its value along a BGP. It satisfies the recursive equation:

$$\widehat{x}_t = \Lambda \Gamma v \gamma E_t \{ \widehat{x}_{t+1} \} + \widehat{y}_t - \frac{\Lambda \Gamma v \gamma}{1 - \Lambda \Gamma v \gamma} \widehat{r}_t$$

Furthermore, log-linearization of (25) about a BGP yields:

$$\widehat{q}_t^B = \Lambda \Gamma E_t \{ \widehat{q}_{t+1}^B \} - q^B \widehat{r}_t \tag{56}$$



In a neighborhood of the *bubbleless BGP*  $q^B = 0$  and  $\Lambda\Gamma v = \beta$ , thus implying the approximate aggregate Euler equation:

$$\widehat{c}_t = \beta\gamma E_t\{\widehat{c}_{t+1}\} + (1 - \beta\gamma)(1 - v\gamma)q_t^B + (1 - \beta\gamma)\widehat{y}_t - \beta\gamma\widehat{r}_t$$

Imposing goods market clearing ( $\widehat{c}_t = \widehat{y}_t$ ) and rearranging terms, we obtain a dynamic IS equation:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} + \Phi q_t^B - \widehat{r}_t$$

where  $\Phi \equiv (1 - \beta\gamma)(1 - v\gamma)/\beta\gamma > 0$  and  $\{q_t^B\}$  must satisfy  $q_t^B = (\beta/v)E_t\{q_{t+1}^B\}$  and  $q_t^B \geq 0$  for all  $t$ .

Note that in a bubbleless equilibrium around the bubbleless BGP:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \widehat{r}_t$$

which is identical to the standard dynamic IS equation of the representative consumer model.

In any *bubbly steady state*,  $q^B = \frac{1}{1-\beta\gamma} - \frac{1}{1-\Lambda\Gamma v\gamma}$ . Combining (55) and (56) yields the Euler equation:

$$\widehat{c}_t = \Lambda\Gamma v\gamma E_t\{\widehat{c}_{t+1}\} + (1 - \beta\gamma)(1 - v\gamma)\widehat{q}_t^B + (1 - \beta\gamma)\widehat{y}_t - \Upsilon v\gamma\widehat{r}_t$$

where  $\Upsilon \equiv \left(1 + \frac{(1-\beta\gamma)(\Lambda\Gamma-1)}{1-\Lambda\Gamma v\gamma}\right) > 1$  and  $\Lambda \in \left(1, \frac{\beta}{\Gamma v}\right)$ . Imposing goods market clearing:

$$\widehat{y}_t = \frac{\Lambda\Gamma v}{\beta} E_t\{\widehat{y}_{t+1}\} + \Phi\widehat{q}_t^B - \frac{\Upsilon v}{\beta}\widehat{r}_t \quad (57)$$

In the particular case of  $\delta = 0$ , we have  $\Lambda\Gamma = \Upsilon = 1$  thus implying a dynamic IS equation of the form:

$$\widehat{y}_t = \frac{v}{\beta} E_t\{\widehat{y}_{t+1}\} + \Phi\widehat{q}_t^B - \frac{v}{\beta}\widehat{r}_t$$

#### 4. Conditions for equilibrium uniqueness around the bubbleless BGP.

The necessary and sufficient conditions for  $\mathbf{A}_T$  to have two eigenvalues within the unit circle are: (i)  $|\det(\mathbf{A}_T)| < 1$  and (ii)  $|tr(\mathbf{A}_T)| < 1 + \det(\mathbf{A}_T)$  (LaSalle (1986)). Note that in the case of a bubbleless BGP,  $\det(\mathbf{A}_T) = \frac{\Phi}{1+\kappa\phi_\pi} > 0$  while  $tr(\mathbf{A}_0) = \frac{1+\kappa+\Phi}{1+\kappa\phi_\pi} > 0$ . Condition (i) requires  $\phi_\pi > (\Phi - 1)/\kappa$ . Condition (ii) corresponds to  $\phi_\pi > 1$ .

## 5. Sunspot Fluctuations

Let the equilibrium be described by the system of difference equations

$$\mathbf{x}_t = \mathbf{A}E_t\{\mathbf{x}_{t+1}\}$$

where  $\mathbf{x}_t \equiv [\widehat{y}_t, \pi_t, \widehat{q}_t^B]'$  are all non-predetermined variables. Let  $\mathbf{A}$  have  $q \leq 3$  eigenvalues with modulus less than one.

Consider the transformation  $\mathbf{x}_t = \mathbf{Q}\mathbf{v}_t$  where  $\mathbf{Q}\mathbf{J}\mathbf{Q}^{-1} = \mathbf{A}$  where  $\mathbf{J}$  is the canonical Jordan matrix and  $\mathbf{Q} \equiv [\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{q}^{(3)}]$  is the matrix of generalized eigenvectors, corresponding to the three eigenvalues. Thus,

$$\mathbf{v}_t = \mathbf{J}E_t\{\mathbf{v}_{t+1}\}$$

where  $\mathbf{J} = \begin{bmatrix} \mathbf{J}_u & 0 \\ 0 & \mathbf{J}_s \end{bmatrix}$  and  $\mathbf{v}_t = [\mathbf{v}_t^u, \mathbf{v}_t^s]'$ .

Consider the case where  $\mathbf{A}$  has two eigenvalues with modulus less than one and one with modulus greater than one. For concreteness, assume  $|\lambda_1| < 1$ ,  $|\lambda_2| < 1$ , and  $|\lambda_3| > 1$ . If all eigenvalues are real  $\mathbf{J}_u = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ ,  $\mathbf{J}_s = \lambda_3$ , and  $\mathbf{q}^{(k)}$  corresponds to the eigenvector associated with eigenvalue  $k$ , for  $k = 1, 2, 3$ . Otherwise, if  $\lambda_1 = a + bi$  and  $\lambda_2 = a - bi$  are complex conjugates,  $\mathbf{J}_u = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ ,  $\mathbf{J}_s = \lambda_3$  and  $\mathbf{q}^{(1)}$  and  $\mathbf{q}^{(2)}$  are, respectively, the imaginary and real components of the eigenvector associated with the complex eigenvalues.

Thus,  $\mathbf{v}_t^u = 0$  all  $t$ . In addition  $\mathbf{v}_t^s = \lambda_3 E_t\{\mathbf{v}_{t+1}^s\}$ , which has a stable solution:

$$\mathbf{v}_t^s = \lambda_3^{-1} \mathbf{v}_{t-1}^s + \xi_t$$

where  $\xi_t$  is a martingale-difference (univariate) process.

Thus, we have  $\mathbf{x}_t = \mathbf{q}^{(3)}\mathbf{v}_t^s$  is the sunspot solution. The three variables in  $\mathbf{x}_t$  will be perfectly correlated (positively or negatively), and will display a first-order autocorrelation  $\lambda_3^{-1}$ . In the simulations reported normalize the the third element of  $\mathbf{q}^{(3)}$  to unity, in which case the innovation in  $\mathbf{v}_t^s$  (denoted by  $\xi_t$ ) corresponds to the innovation in the bubble.

Consider next the case where  $\mathbf{A}$  has one eigenvalue with modulus less than one and two with modulus greater than one. For concreteness, assume  $|\lambda_1| < 1$ ,  $|\lambda_2| > 1$ , and  $|\lambda_3| > 1$ . If all eigenvalues are real  $\mathbf{J}_u = \lambda_1$ ,  $\mathbf{J}_s = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix}$  and  $\mathbf{q}^{(k)}$  corresponds to the eigenvector associated with eigenvalue  $k$ , for  $k = 1, 2, 3$ . Otherwise, if  $\lambda_2 = a + bi$  and  $\lambda_3 = a - bi$  are complex conjugates,  $\mathbf{J}_u = \lambda_1$ ,  $\mathbf{J}_s = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  and  $\mathbf{q}^{(2)}$  and  $\mathbf{q}^{(3)}$  are, respectively, the imaginary and real components

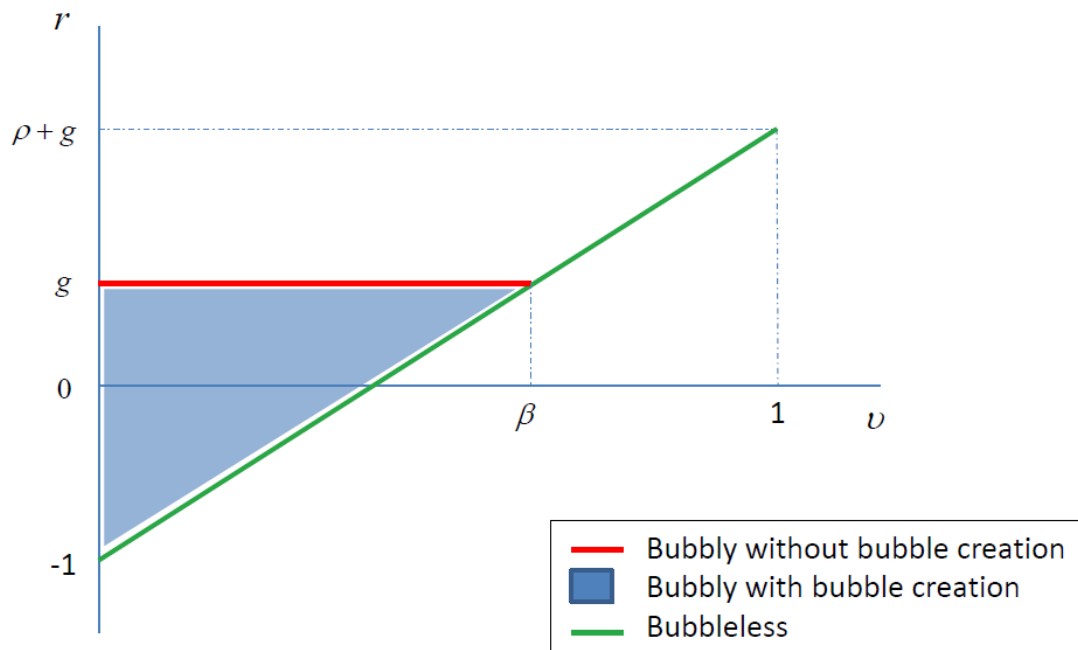
of the eigenvector associated with the complex eigenvalues, while  $\mathbf{q}^{(1)}$  is the eigenvector associated with  $\lambda_1$ .

Thus,  $\mathbf{v}_t^u = 0$  all  $t$ . In addition  $\mathbf{v}_t^s = \mathbf{J}_s E_t\{\mathbf{v}_{t+1}^s\}$ , which has a stable solution:

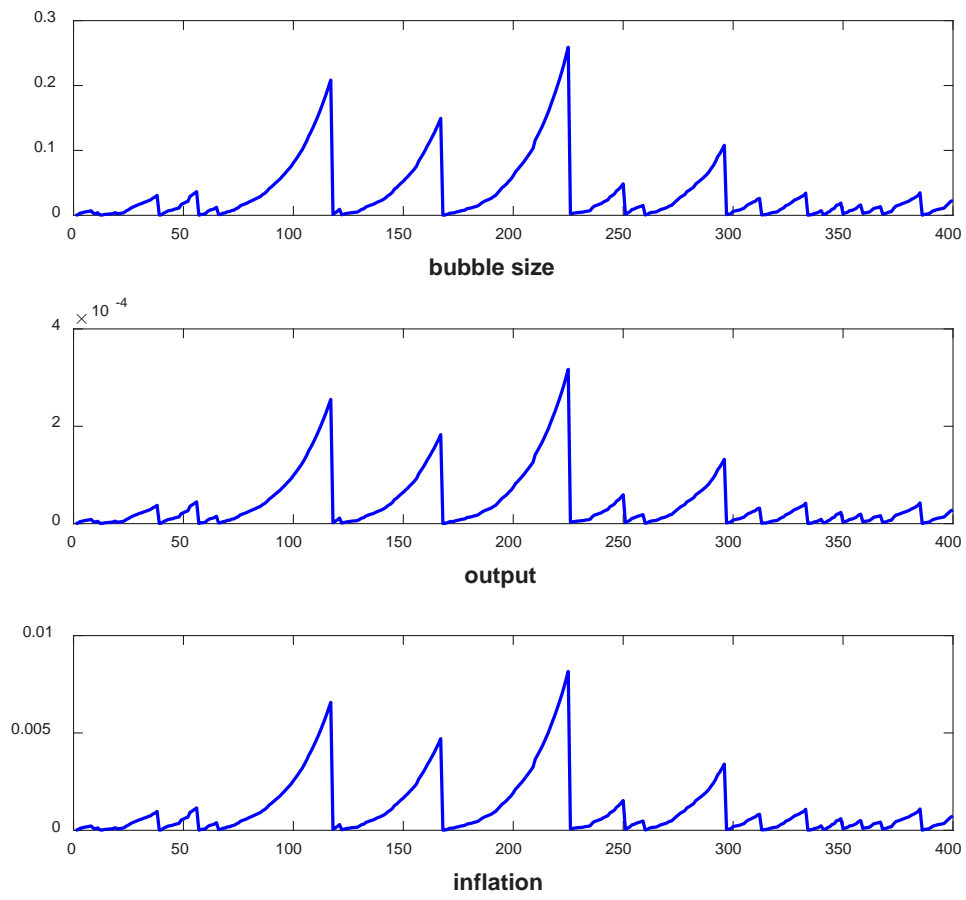
$$\mathbf{v}_t^s = \mathbf{J}_s^{-1} \mathbf{v}_{t-1}^s + \boldsymbol{\xi}_t$$

where  $\boldsymbol{\xi}_t$  is a (bivariate) martingale-difference process.

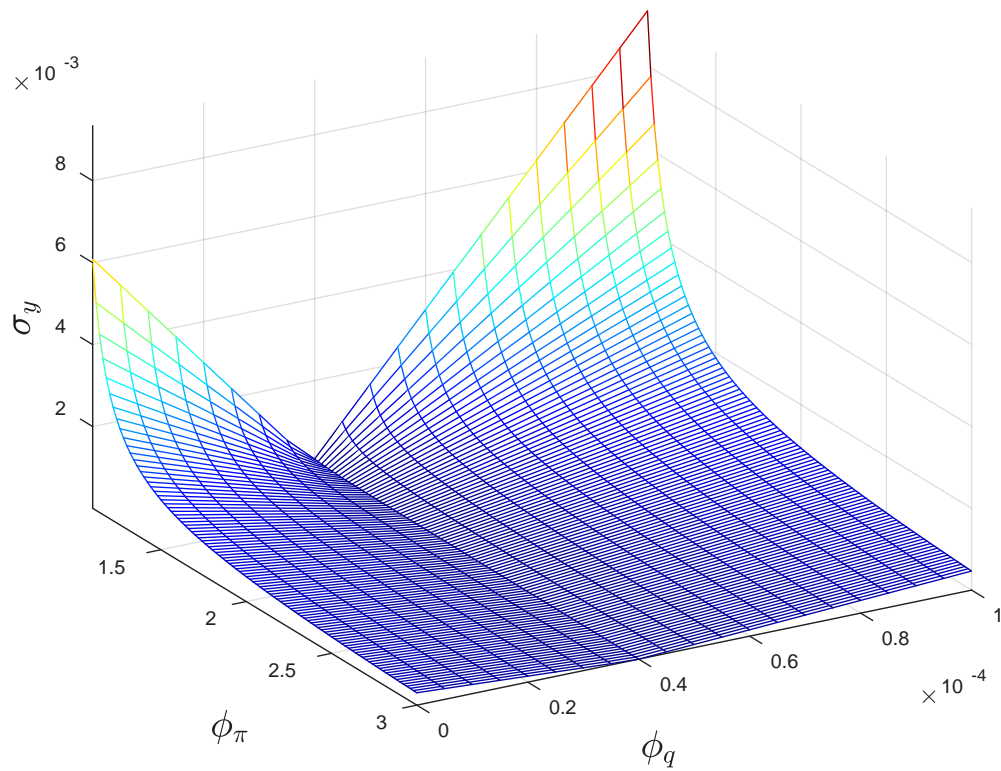
Thus, we have  $\mathbf{x}_t = [\mathbf{q}^{(2)}, \mathbf{q}^{(3)}] \mathbf{v}_t^s$  is the sunspot solution. We can normalize the third element of  $\mathbf{q}^{(2)}$  and  $\mathbf{q}^{(3)}$  to unity.



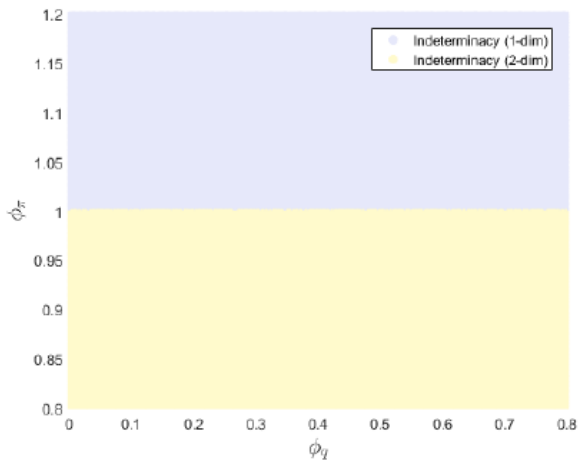
**Figure 1. Balanced Growth Paths**



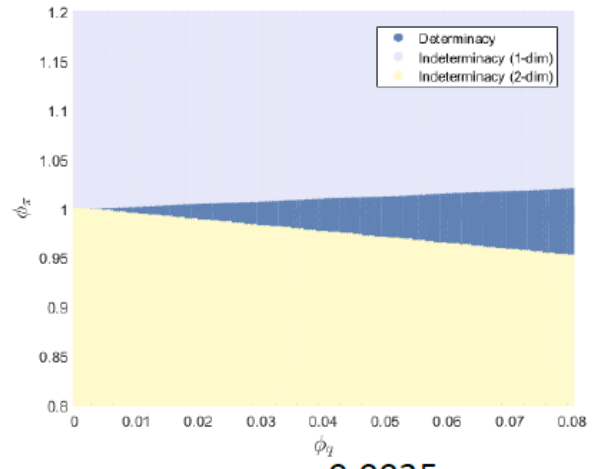
**Figure 2. Simulated Bubble-Driven Fluctuations  
around the Bubbleless BGP**



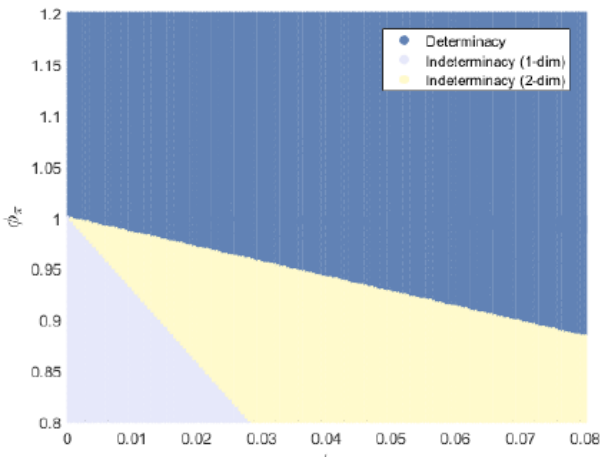
**Figure 3. Bubble-driven Fluctuations: Monetary Policy and Macro Volatility in a Neighborhood of the Bubbleless BGP**



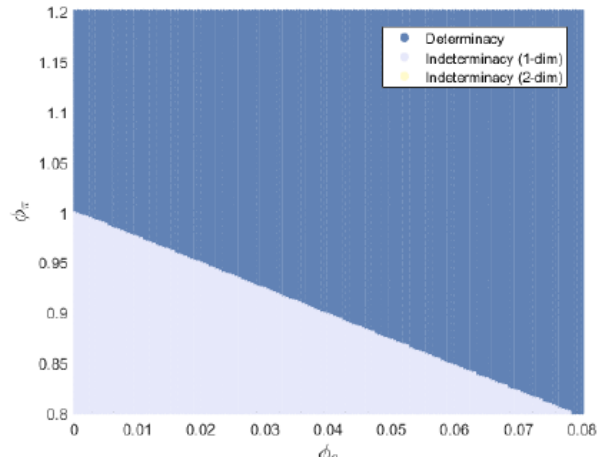
$r = 0.00335$



$r = 0.0035$

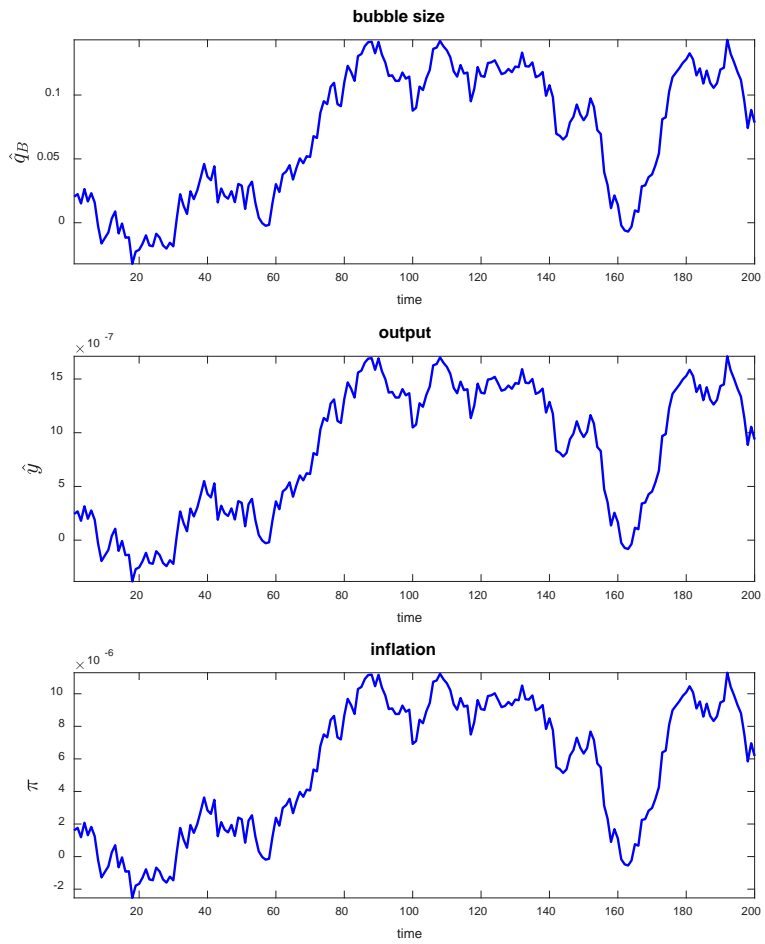


$r = 0.0037$



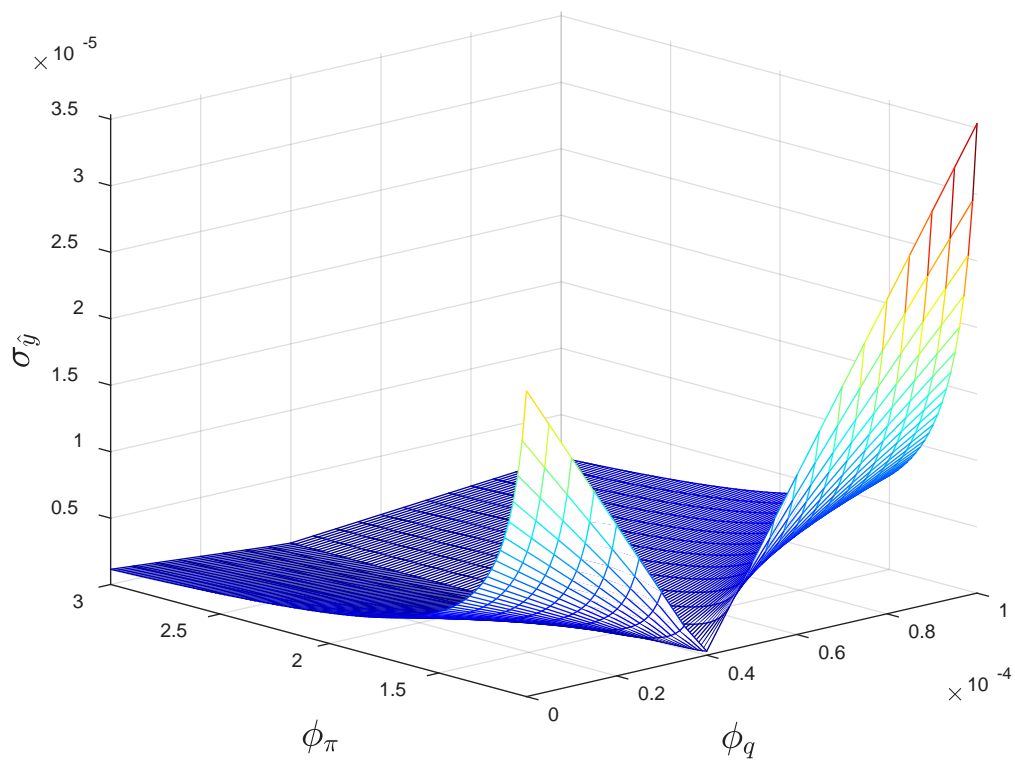
$r = 0.004$

**Figure 4. Determinacy and Indeterminacy Regions around Bubbly BGPs**

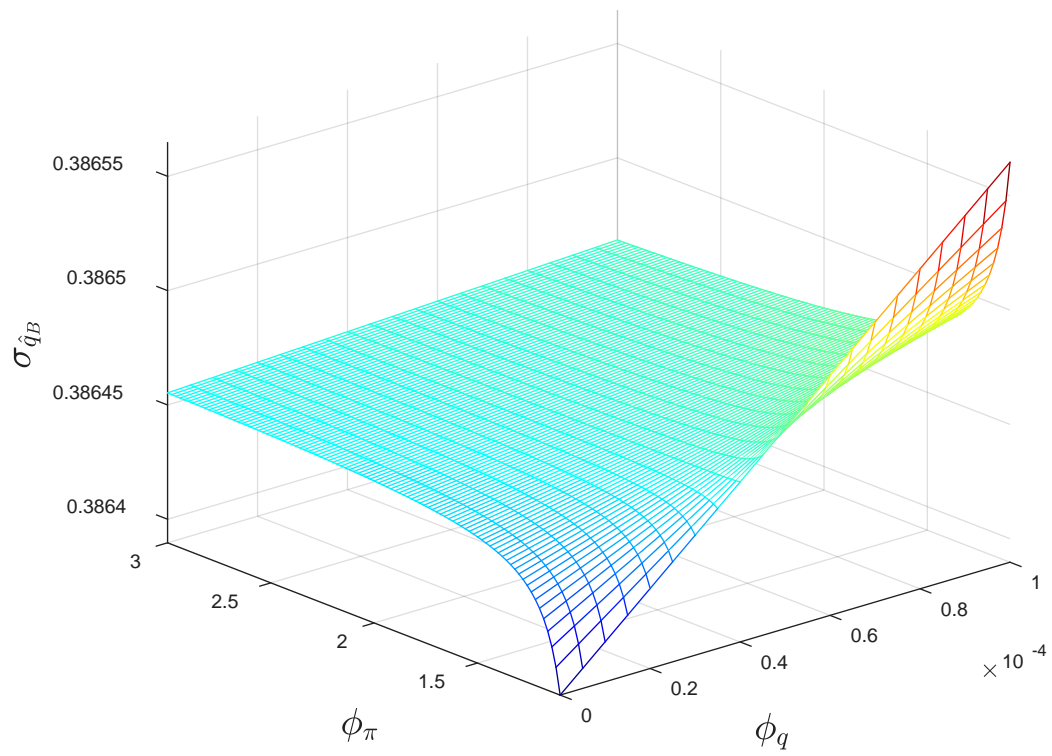


**Figure 5. Simulated Bubble-driven Fluctuations around a Bubbly BGP**





**Figure 6.a Bubble-driven Fluctuations: Monetary Policy and Output Volatility in a Neighborhood of a Bubbly BGP**



**Figure 6.b Bubble-driven Fluctuations: Monetary Policy and Bubble Volatility in a Neighborhood of a Bubbly BGP**