Long Term Government Bonds*

Faraglia, Elisa [†] Marcet, Albert [‡] Oikonomou, Rigas [§] Scott, Andrew [¶]

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Abstract

We study how the issuance of long bonds affects optimal fiscal policy. Long bonds are usually modelled as having two features that are *not* found in the data: a) zero coupons and b) previously issued bonds are repurchased each period regardless of their time to maturity. The literature has found that under a) and b) issuing long bonds provides fiscal insurance. We show that these assumptions are not innocuous. Specifically we find that long bonds may not complete the markets even in the absence of uncertainty and under certain assumptions (namely those that are most empirically relevant) long bonds introduce additional tax volatility. This may offset the attractiveness that long bonds provide through fiscal insurance, especially after a period of very high deficits such as a war or financial recession. Introducing coupons alleviates the additional tax volatility but only partially and does so by reducing the ability of long bonds to provide insurance. Our focus on long bonds also forces us to consider issues of commitment.

We show that the role of commitment under incomplete markets is that the government promises future tax changes in order to reduce current funding costs (interest rate twisting). This introduces further additional tax volatility at different frequencies. If we remove assumptions a) and b) interest rate twisting takes a very different form, showing again that those assumptions matter. We propose an alternative to full commitment that eliminates interest rate twisting, dramatically reduces both the state space and computational cost in a way that has relevance for a wide class of models.

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Corresponding Author: Rigas Oikonomou, IRES-CORE, Université Catholique de Louvain, Collège L. H. Dupriez, 3 Place Montesquieu 1348 Louvain la Neuve, Belgium.

[†]University of Cambridge and CEPR.

[‡]MOVE, ICREA, Institut d'Anàlisi Econòmica-CSIC, Barcelona GSE, UAB, and CEPR.

[§]Université Catholique de Louvain. Email: Rigas.Oikonomou@uclouvain.be

[¶]London Business School and CEPR.

1 Introduction

We study optimal fiscal policy when the government issues real riskless long bonds. Dynamic equilibrium macro models under incomplete markets require specific modelling assumptions about the type of government bonds available. The vast literature on DSGE models considers mostly the case of one-period bonds. The same holds for the literature on optimal policy, as in Barro (1979) and Aiyagari, Marcet, Sargent and Seppälä (2002). This focus on short bonds is surprising given that the share of U.S. government bonds issued with maturity longer than one year is 64% on average.[1] As we show, when long bonds are introduced a number of additional assumptions on bond policy have to be made compared to the case of one period bonds and we find that these issues matter for the model outcome. Introducing long term bonds involves modelling the maturity of these bonds and raises issues regarding when bonds are repurchased and the paying of coupons.

Most papers using long bonds assume zero-coupon payments and a full repurchase of previously issued bonds each period regardless of outstanding maturity. This is the case for most of the work on optimal debt management as in Angeletos (2002), Barro (2003), Buera and Nicolini (2004) in a complete market setting, or Nosbusch (2008), Lustig, Sleet and Yeltekin (2009) in an incomplete market setting. Other papers make simplifying assumptions aimed at making the model solution easier. For instance, a number of papers model long bonds as perpetuities that pay geometrically declining coupons - Woodford (2001), Broner, Lorenzoni and Schmulker (2013), Arellano and Ramanarayanan (2008), Chen, Curdia and Ferrero (2012), Conesa and Kehoe (2015) and Debortoli, Nunes and Yared (2015). Whilst tractable these assumptions are not consistent with observed long run debt instruments: most government bonds are not perpetuities and they always pay constant coupons.

In this paper we consider the impact on the optimal properties of long bonds by varying modelling assumptions around commitment, repurchasing bonds and coupon payments and find that doing so leads to non-trivial variations and increases in tax volatility. We argue that some features of long bonds that have been usually ignored in the literature are important in order to explain actual bond issuance and its virtues. The existing literature has stressed the advantages of issuing long bonds that arise from fiscal insurance (essentially the covariance of long bond prices with government expenditure shocks) and their ability to achieve full insurance and complete markets. By contrast we show that long bonds without repurchase cannot complete the market, *even* under certainty. More generally, we find that the advantages of fiscal insurance that long bonds provide may be offset by additional tax volatility. These limitations of long bonds are strongest when the assumptions we make are closest to actual practice, namely when long bonds are not repurchased each and every period and when coupons are fixed for the duration of the bond.[2]

Further the literature to date suggests that a sufficient set of measurability conditions for implementability constraints under incomplete markets is to require that the value of government debt equals future discounted surpluses in each period. However we find that in the case of no-buyback, if coupons are low, an additional boundedness condition is required for a sufficient set of measurability conditions. This additional condition reflects the fact that long bonds under no-buyback may gener-

[1]This figure is for the period 1955-2011, taken from Faraglia, Marcet, Oikonomou and Scott (2014b).

[2]Faraglia, Marcet, Oikonomou and Scott (2014b) document that most US government bonds are not repurchased or they are repurchased close to maturity date.

ate additional tax volatility. We show that this additional tax volatility is not large during standard business cycle fluctuations but it is very substantial in the case of a sequence of very large deficits due, for example, to a war or a financial recession.

Throughout our analysis we take for granted the type of bonds that exist but introduce features that are present in actually issued bonds. Most of the existing literature (including many papers in our bibliography) takes a similar approach, studying optimal policy when the set of financial instruments available to the government is given, with the set of instruments considered justified implicitly by similarity with instruments actually used by governments. Whether issuing a given type of bond can be justified as a response to market imperfections is in our view an important research agenda albeit one we do not pursue here.[3] Our approach instead is to examine the role of debt management and optimal fiscal policy under differing assumptions that arise when one models long bonds as in the real world. Both through simulations and analytic examples we show that long bonds potentially affect optimal policy in three ways - fiscal insurance, interest rate twisting and rollover risk. Under previously used assumptions only the first channel is noticeable and long bonds dominate short bonds in debt management. Under no repurchase the other effects become important and lessen the appeal of long bonds.

The plan of the paper is as follows. We start in Section 2 with the usual assumptions about long bonds found in the literature, namely, zero coupon payments and full repurchase each period. We show how the government has an incentive to twist interest rates by committing to vary tax rates at the redemption date in order to minimise funding costs. Whilst the fiscal insurance properties of long bonds highlighted by Angeletos (2002), Barro (2003), Buera and Nicolini (2004) helps reduce tax volatility this interest rate twisting effect increases their volatility. In the usual case where only one period debt is considered this effect is conflated with the usual impact effect on taxes and is not observed. By focusing on a long bond we disentangle the impact effect on taxes from this intertemporal effect that occurs around redemption.

In Section 3 we consider the role of commitment and time consistency in this model with long term debt. Denote the bond maturity date as N. A common numerical approach to solve for models of optimal taxation involving long bonds is to use a recursive solution that introduces as state variables N lags of the Lagrange multipliers λ attached to the government's intertemporal budget constraint. This makes solving models with long bonds computationally challenging as the state space quickly becomes unwieldy with long maturities. From our reading of the debt management literature it is unclear why these Lagrange multipliers are needed or how they influence optimal policy and neither is there any explicit discussion of the role of commitment (with the notable exception of Lucas and Stokey (1983) and Debortoli, Nunes and Yared (2015)). We show these two lacunae are related - the role of the co-state variables λ is to enforce in the appropriate continuation problem the promises for future taxes that drive optimal interest rate twisting. Having identified the channel through which the state space quickly becomes cumbersome in the presence of long bonds we can then modify the model set up to alleviate this problem. We do this by proposing a model of independent powers (IP) where the government sets taxes but takes interest rates as given. This dramatically reduces the size of the

[3] Intuitively a number of justifications for not repurchasing spring to mind and are often cited by Debt Management Offices - transaction costs, rollover risks, market disturbances due to large scale government interventions, moral hazard and asymetric information. A theory of market turnover in the secondary market would be useful - a substantive agenda not just for the debt management literature but the whole of finance.

state vector and removes the interest rate twisting channel, hence comparing the full commitment solution with independent power solution throughout the paper allows us to demonstrate the role of commitment.

In Section 4 we perform simulations, we examine the magnitude of interest rate twisting and, by comparing the full commitment model with our model of independent powers we show how commitment introduces additional tax volatility.

In Section 5 we move from the standard assumption of full repurchase each period regardless of maturity to the opposite assumption that every bond once issued is only repurchased at its scheduled redemption date. As most of the literature we keep the assumption of zero coupons in this section. This assumption influences allocations, first, because under incomplete markets the timing of cash flows matter. Second, because no early buyback induces additional rollover cycles in taxes with the same periodicity as the maturity of debt. A large deficit in t drives $b_{N,t}$ upwards due to a standard buffer stock effect, this increases future interest payments. But higher future interests are only paid in periods $t+N$, $t+2N$, ..., thus taxes in these periods are much higher than in the interim periods. This introduces additional tax volatility, a one-period bond would spread interest payments over all periods $t, t+1, t+2, ...$ A simple example shows that, even under certainty, a long bond does not complete the markets and introduces tax volatility. Not only are taxes more volatile but debt displays more complicated dynamics than the martingale property documented by Aiyagari et al (2002). We find that this additional tax volatility is not significant when we calibrate shocks to standard business cycle fluctuations but it is very high when we calibrate initial conditions as they would have been at the end of WWII if only long bonds were available. In this sense long bonds generate greater tax volatility than short bonds, introducing a trade off between fiscal insurance and roll over cycles.

In Section 6 we extend our analysis to positive coupons. Introducing coupons enables us to introduce duration issues into our analysis of debt management. Duration reflects the average time over which the cash payments associated with a bond are paid. This is important both because the timing of cash flows matter under incomplete markets and because the responsiveness of bond prices to shocks is directly proportional to the duration of the bond. Under the standard assumptions in the literature, namely under short bonds or long bonds with full repurchase, there is no distinction between duration and maturity. With long bonds, no buy back and coupons duration can vary for a given maturity. Some analytic examples show how long bonds without repurchase do offer better tax smoothing opportunities if coupons are sufficiently large, therefore alleviating the rollover cycles emphasized in Section 5. We show how results on measurability constraints in Aiyagari et al. (2002) and Angeletos (2002) can be generalized to the case of sufficiently high coupons, but they do not hold under low coupons. Our numerical results also show that (maintaining no repurchase) the introduction of fixed coupon bonds helps reduce tax volatility. For bonds of maturity N, the higher N the more introducing coupons helps to reduce tax volatility and lessens the impact of $N-$period rollover cycles. However whilst coupons help reduce N period volatility they shorten the duration of a bond and so reduce the effectiveness of long bonds in achieving fiscal insurance. Even under coupons we find that tax volatility after a war is very high under no repurchase.

A final section concludes. An appendix considers some analytical details as well as comparing our fully specified approach to modelling long bonds with an approximation based on perpetuities which pay geometrically declining coupons.

2 Interest Rate Twisting

In this section we outline our base model, in essence an extension of Aiyagari et al. (2002) to the case of a riskless real bond of maturity N where $N > 1$. We start with the standard modelling assumptions of zero-coupon bonds and that the government buys back all previously issued bonds each period regardless of maturity.

2.1 The Base Model

We assume the economy produces a single non-storable good with technology

$$c_t + g_t \leq A - x_t, \tag{1}$$

for all t, where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively. The exogenous stochastic process g_t is the only source of uncertainty. The consumer is endowed with A units of time that she allocates between leisure and labour. The representative consumer has utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t\right) + v\left(x_t\right) \right\} \tag{2}$$

and faces a proportional tax rate τ_t on labor income. The representative firm maximizes profits and both consumers and firms act competitively by taking prices and taxes as given. Consumers, firms and government all have full information, i.e. they observe all shocks up to the current period, and all variables dated t are chosen contingent on histories $g^t = (g_t, \ldots, g_0)$. All agents, including the government, have rational expectations.

Agents can only borrow and lend in the form of a zero-coupon, risk-free, N-period bond so that the government budget constraint is:

$$g_t + p_{N-1,t} b_{N,t-1} = \tau_t \left(A - x_t\right) + p_{N,t} b_{N,t} \tag{3}$$

where $b_{N,t}$ denotes the number of bonds the government issues at time t. Each bond pays one unit of consumption good in N periods time with complete certainty. The price of an i-period bond at time t is $p_{i,t}$. As is standard in the literature on long bonds, we assume that at the end of each period the government buys back the existing stock of debt and then reissues new debt of maturity N, these repurchases are reflected in the left side of the budget constraint (3). In addition, government bonds have to remain within upper and lower limits $\underline{M}$ and $\overline{M}$ so as to rule out Ponzi schemes:[4]

$$\underline{M} \leq \beta^N b_{N,t} \leq \overline{M}. \tag{4}$$

The term β^N in this constraint reflects the value of the long bond at steady state so that the limits $\underline{M}$, $\overline{M}$ appropriately refer to the value of debt and are comparable across maturities.[5]

[4] Similar debt constraints are assumed in Aiyagari et al (2002).

[5] Obviously the actual value of debt is $p_{N,t} b_{N,t}$, we substitute $p_{N,t}$ by its steady state value β^N in this constraint

We assume that after purchasing a long bond the household entertains only two possibilities: one is to resell the government bond in the secondary market in the period immediately after having purchased it, the other possibility is to hold the bond until maturity.[6] Letting $s_{N,t}$ be the sales in the secondary market the household's problem is to choose stochastic processes $\{c_t, x_t, s_{N,t}, b_{N,t}\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints:

$$c_t + p_{N,t}b_{N,t} = (1-\tau_t)(A-x_t) + p_{N-1,t}s_{N,t} + b_{N,t-N} - s_{N,t-N+1}$$

with prices and taxes $\{p_{N,t}, p_{N-1,t}, \tau_t\}$ taken as given. The household also faces debt limits analogous to (4). We assume for simplicity that these limits are less stringent than those faced by the government, so that in equilibrium the household's problem always has an interior solution.

The consumer's first order conditions of optimality are given by

$$\frac{v_{x,t}}{u_{c,t}} = 1-\tau_t \tag{5}$$

$$p_{N,t} = \frac{\beta^N E_t(u_{c,t+N})}{u_{c,t}} \tag{6}$$

$$p_{N-1,t} = \frac{\beta^{N-1} E_t(u_{c,t+N-1})}{u_{c,t}} \tag{7}$$

where $u_{c,t} \equiv u'(c_t)$ and $v_{x,t} = v'(x_t)$.

2.1.1 The Ramsey problem

We follow a standard definition of Ramsey equilibrium, assuming the government has full commitment to implement the best sequence of (possibly time inconsistent) taxes and government debt knowing equilibrium relationships between prices, taxes and allocations. Using (5), (6) and (7) to substitute for taxes and consumption the Ramsey equilibrium can be found by solving

$$\max_{\{c_t, b_{N,t}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \tag{8}$$

$$\text{s.t.} \qquad \beta^{N-1}E_t(u_{c,t+N-1})\, b_{N,t-1} = S_t + \beta^N E_t(u_{c,t+N})\, b_{N,t} \tag{9}$$

and (4) with x_t implicitly defined by (1). $S_t = (u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t}g_t$ is the "discounted" surplus of the government.

We set up the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + v(x_t) + \lambda_t \left[S_t + \beta^N u_{c,t+N} b_{N,t} - \beta^{N-1} u_{c,t+N-1} b_{N,t-1} \right]$$
$$+ \nu_{1,t}\left(\overline{M} - \beta^N b_{N,t}\right) + \nu_{2,t}\left(\beta^N b_{N,t} - \underline{M}\right) \Big\}$$

where λ_t is the Lagrange multiplier associated with the government budget constraint, i.e. the excess burden of taxation, and $\nu_{1,t}$ and $\nu_{2,t}$ are the multipliers associated with the debt limits.

for simplicity.

[6] We need to introduce secondary market sales $s_{N,t}$ in order to price the repurchase of the bond.

The first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t}\left(c_t + g_t\right) - v_{x,t}\right) + u_{cc,t}\left(\lambda_{t-N} - \lambda_{t-N+1}\right) b_{N,t-N} = 0 \tag{10}$$

$$E_t\left(u_{c,t+N}\lambda_{t+1}\right) = \lambda_t E_t\left(u_{c,t+N}\right) + \nu_{2,t} - \nu_{1,t} \tag{11}$$

for all $t = 0, 1, \ldots,$ with $\lambda_{-1} = \ldots = \lambda_{-N} = 0$.

Assuming g_t is a Markov process, Corollary 3.1 in Marcet and Marimon (2014) implies the solution satisfies the recursive structure:

$$\begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} = F(g_t, \lambda_{t-1}, \ldots, \lambda_{t-N}, b_{N,t-1}, \ldots, b_{N,t-N}) \tag{12}$$

$$\lambda_{-1} = \ldots = \lambda_{-N} = 0, \quad \text{given } b_{N,-1} \tag{13}$$

for a time-invariant policy function F. Therefore the state vector in this recursive formulation has dimension $2N + 1$.[7]

These FOCs help characterize some features of optimal fiscal policy with long bonds. Following the discussion in Aiyagari et al. (2002) we see that, in the case where debt limits are non binding, i.e. for t such that $\nu_{1,t} = \nu_{2,t} = 0$, (11) implies λ_t is a risk-adjusted martingale, with risk-adjustment measure $\frac{u_{c,t+N}}{E_t\left(u_{c,t+N}\right)}$, indicating that the presence of the state variable λ in the policy function imparts persistence in the variables of the model.

The term

$$\mathcal{D}_t = \left(\lambda_{t-N} - \lambda_{t-N+1}\right) b_{N,t-N} \tag{14}$$

in (10) is key for our analysis of long bonds and interest rate twisting as it captures the feature that what happened in period $t - N$ has a specific impact on today's taxes. In particular, as we shall see, this term captures the fact that governments when they issue debt at $t - N$ make (time inconsistent) commitments to influence future taxes in order to affect the interest rate payable on N period debt.

Before outlining some analytical insights consider the following intuition. Since in the first best we have $u_{c,t} - v_{x,t} = 0$ and zero taxes, this suggests that the higher is $\mathcal{D}_t$ the further the model is pulled away from the first best and taxes are higher. Thus when the term $\mathcal{D}_t$ is positive it can be thought of as introducing a higher distortion in a given period. In periods when g_{t-N+1} is very high we have that the cost of the budget constraint is high so λ_{t-N+1} is high, and if the government is in debt $\mathcal{D}_t < 0$ and optimal policy is to lower taxes t. Of course this is not a tight argument, as λ_t also responds to the shocks that have happened between t and $t - N$ and λ_t also plays a role in (10), but this argument is at the core of the interest rate twisting policy we identify below.

[7] This allows for a simpler recursive formulation than the promised utility approach, as the co-state variables λ do not have to be restricted to belong to the set of feasible continuation variables so that the continuation problem is well defined. In Section 3.1 we show this continuation problem explicitly.

2.2 Analytic Results

2.2.1 A model under certainty

Assume for now that government spending is constant, $g_t = \overline{g}$. In this case long bonds complete the market so that the standard result ensures that all equilibrium constraints are summarized in a single implementability constraint, namely

$$\sum_{t=0}^{\infty} \beta^t \frac{S_t}{u_{c,0}} = b_{N,-1} \ \ p_0^{N-1} \tag{15}$$

rewritten as

$$\sum_{t=0}^{\infty} \beta^t \ \ S_t = b_{N,-1} \ \ \beta^{N-1} u_{c,N-1}.$$

Consider the case when the government is initially in debt such that $b_{N,-1} > 0$. It is clear that the funding costs of initial debt $b_{-1}^N > 0$ can be reduced by manipulating consumption so as to achieve $c_t < c_{_{N-1}}$ for all $t \neq N$, as this lowers the total cost of initial debt on the right side of this equation. As long as the elasticity of consumption with respect to wages is positive, which would be the case for empirically reasonable calibrations, higher $c_{_{N-1}}$ will be achieved by promising a tax cut in period $N-1$ relative to other periods. In other words, the planner sets

$$\begin{aligned} \tau_t &= \overline{\tau} \text{ for all } t \neq N-1 \\ \overline{\tau} &> \tau_{_{N-1}}. \end{aligned} \tag{16}$$

This promise achieves a reduction of $u_{c,N-1}$ and so reduces the cost of outstanding debt by twisting the long end of the yield curve downwards. This is the same interest rate manipulation channel noted by Lucas and Stokey (1983) except here it is shifted N periods forward due to the maturity of bonds. Note that even though there are no fluctuations in this economy, (16) shows that the optimal policy implies that the government desires to introduce variability in taxes.

2.2.2 A model with uncertainty at $t = 1$

We now introduce uncertainty into our model, although in the interest of obtaining analytic results, only in the first period, i.e. g is given by[8]:

$$\begin{cases} g_t = \overline{g} & \text{for } t = 0 \text{ and } t \geq 2 \\ g_1 \sim F_g & \end{cases}$$

for some non-degenerate distribution F_g.

This is a special case of the model in Section 2.1 so the FOCs derived there apply. Since there is no more uncertainty for $t > 1$ we have $E_t(\lambda_{t+1}) = \lambda_{t+1}$ for all $t \geq 1$, so the martingale condition (11) implies $\lambda_{t+1} u_{c,t+N} = \lambda_t u_{c,t+N}$ and

$$\lambda_t = \lambda_1 \qquad t > 1. \tag{17}$$

[8]The analytics of this economy are similar to those of Nosbusch (2008), except that this is an infinitely lived economy so debt is not cancelled in period $t = 2$, but stays constant.

Therefore, in the case of short bonds ($N = 1$), (10) and feasibility imply c_t and τ_t constant for $t \geq 2$ reflecting the fact that even though markets are incomplete the government smooths taxes after the shock is realized. However, clearly c_1 and τ_1 will be a function of the realization of g_1.

For the case of long bonds when $N > 1$, letting $\mathcal{D}_t = (\lambda_{t-N} - \lambda_{t-N+1})\, b_{N,t-N}$ the FOC with respect to consumption (10) is satisfied for

$$\mathcal{D}_t = 0 \quad \text{for } t \geq 0 \text{ and } t \neq N-1, N \tag{18}$$

$$\mathcal{D}_{N-1} = -\lambda_0 b_{N,-1}\ , \quad \mathcal{D}_N = (\lambda_0 - \lambda_1)\, b_{N,0}. \tag{19}$$

Combining this with feasibility, (17) and the fact that $g_2 = \overline{g}$ for all $t \geq 2$ means that equilibrium satisfies

$$c_t = \quad c^*(g_1) \quad \text{for all } t \geq 2 \text{ and } t \neq N, N-1 \tag{20}$$

for a certain function c^* i.e. consumption is the same in all periods $t \geq 2$ except $t = N, N-1$.

In this model, when the shock g_1 is realised the government optimally spreads out the taxation cost of this shock over current and future periods. Typically the government gets in debt in period 1 if g_1 is high, so all future taxes for $t \geq 2$ are higher and future consumption lower. This would also happen with short bonds $N = 1$. What is new with long bonds is that optimal policy introduces an additional source of tax volatility, since taxes vary in periods $N-1$ and N, even though by the time the economy arrives at these periods no more shocks have occurred for a long time.

To make this argument precise consider the utility function

$$\frac{c_t^{1-\gamma_c}}{1-\gamma_c} - B\frac{(1-x_t)^{1+\gamma_l}}{1+\gamma_l} \tag{21}$$

for $\gamma_c, \gamma_l, B > 0$, and $A = 1$.

Result 1. *Assume utility (21) and $b_{N,-1} > 0$.* Then

$$\tau_1 = \tau_t \text{ for all } t \geq 1, \;\; t \neq N-1, N. \tag{22}$$

Furthermore, for a high enough realization of g_1 we have

$$\tau_1 > \tau_{N-1}, \tau_N. \tag{23}$$

The inequalities are reversed if $b_{N,-1} < 0$ or if the realization of g_1 is sufficiently low.

Proof.

Towards (22) note first that from (20) we have $\tau_t = \tau_2$ for all $t \geq 2$ and $t \neq N-1, N$.
(10) and (17) give

$$\frac{u_{c,t}}{v_{x,t}} - \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1)\, B} + (\lambda_{t-N} - \lambda_{t-N+1})\, \mathcal{F}_t = 0 \quad \text{for } t \geq 1$$

where $\mathcal{F}_t \equiv \frac{u_{cc,t} b_{N,t-N}}{(1+(1-\gamma_c)\lambda_1)B}$. Consider $t = 1$. For any long maturity $N > 1$ we have that $\lambda_{t-N} = \lambda_{t-N+1} = 0$ when $t = 1$ so that

$$\frac{u_{c,1}}{v_{x,1}} = \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1)\, B}. \tag{24}$$

Therefore we can write

$$\frac{u_{c,t}}{v_{x,t}} - \frac{u_{c,1}}{v_{x,1}} = (\lambda_{t-N+1} - \lambda_{t-N})\,\mathcal{F}_t = 0 \quad \text{for } t \geq 1. \tag{25}$$

For $N > 1$ and from (13) we have $\lambda_{t-N+1} = \lambda_{t-N} = 0$ when $t = 2$. This and (25) gives $\tau_1 = \tau_2$ so that we have (22).

Towards (23) we now show that $\mathcal{F}_t < 0$ for $t = N-1, N$. Since $\lambda_1, B, \gamma_l > 0$ we have that $B + (\gamma_l + 1)\lambda_1 > 0$. Since $u_{c,1}, v_{x,1} > 0$ clearly (24) implies that $(1 + (-\gamma_c + 1)\lambda_1)\,B > 0$. Since we consider the case of initial government debt $b_{N,-1} > 0$ this leads to $b_{N,0} > 0$ and since $u_{cc,1} < 0$ we have $\mathcal{F}_t < 0$ for $t = N-1, N$.

For $t = N-1$ we have $\lambda_{t-N} - \lambda_{t-N+1} = -\lambda_0 < 0$ it follows

$$\frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \implies \tau_{N-1} < \tau_t \text{ for all } t > 1, \quad t \neq N-1, N.$$

Also, it is clear from (24) that high g_1 implies a high λ_1. Since the martingale condition implies $E_t\,(u_{c,N}\lambda_1) = \lambda_0 E_0\,(u_{c,N})$ for higher than average g_1 we have $\lambda_1 > \lambda_0$ Therefore, for $t = N$ and g_1 high enough we have $\lambda_{t-N} - \lambda_{t-N+1} = \lambda_0 - \lambda_1 < 0$ so that (25) implies

$$\frac{u_{c,N}}{v_{x,N}}, \frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \implies \tau_N, \tau_{N-1} < \tau_1. \blacksquare$$

Intuitively, in period $t = N-1$ there is a tax cut for the same reasons as in Section 2.2.1. New in this section is the tax cut (for high g_1) at $t = N$. The intuition for this is clear: when an adverse shock to spending occurs at $t = 1$ the government uses debt as a buffer so $b_{N,1} > b_{N,0}$. This use of debt as a buffer is typical of incomplete market models as it allows tax smoothing by financing part of the adverse shock with higher future taxes. But since future surpluses are higher than expected as the higher interest payments have to be serviced, the government can lower the cost of existing debt by announcing a tax cut in period N, since this will reduce the price $p_{N-1,0}$ of period $t = 1$ outstanding bonds $b_{N,0}$. The tax cut at $t = N$ is a stochastic analog of the tax cut described in Section 2.2.1.

The above result shows that in this model tax policy is not independent of the maturity of government debt. In models of optimal policy the government usually desires to smooth taxes. Taxes would be constant in the above model if the government had access to complete markets. But we find that the government increases tax volatility in period N, long after the economy has received any shock. It is clear from this discussion that what will matter for the policy function is the term $\mathcal{D}_N = (\lambda_0 - \lambda_1)b_{N,0}$ which captures the government's commitment to alter future tax and interest rates. Therefore it is the interaction between past λ's and past b's that determines the size and the sign of today's tax cut.

To summarize, under incomplete markets and in the presence of an adverse shock to spending in period t the government has to take three actions: *i)* increase taxes permanently, *ii)* increase debt permanently, *iii)* announce a tax cut around the time when the outstanding debt matures, namely at $t + N$. Effects *i)* and *ii)* are well known in the literature of optimal taxation under incomplete markets, effect *iii)* is clearly seen in this model with long bonds since the promise is made N periods ahead. Obviously in the case of short maturity $N = 1$ of Aiyagari et al. (2002) the effect of $\mathcal{D}_1$ would be felt in deciding optimally τ_1 but would be confounded with the fact g_1 is stochastic and influences demand for the consumption good. However when $N > 1$ the two effects are disentangled and we see how debt maturity introduces additional dynamics into taxes - *i)* reflects the usual increase in the excess burden of taxation given the adverse fiscal shocks, *ii)* is the usual incomplete market result that says the excess burden should follow a risk adjusted martingale leaving debt to fluctuate whereas *iii)* captures a distinct interest rate twisting channel due to debt maturity whereby governments induce additional tax volatility to reduce funding costs.

As mentioned earlier Lucas and Stokey (1983) also identify this interest rate twisting channel in their discussion of maturity. However because ours is an incomplete market model we identify this as a factor during *all* periods and not just the initial period, and because we have long bonds the interest rate twisting influences consumption N periods ahead.

It is also worth distinguishing this channel, which focuses on *real* interest rates and how future tax commitments influence current interest rates, from a number of related results in the literature that rely on nominal debt and the role of inflation surprises. Chari et al (1991) show how inflation surprises can bring about fluctuations in ex post real interest rates so as to achieve the complete market outcome and Schmitt-Grohe and Uribe (2004) and Siu (2004) extend this case to consider how this role is affected by introducing distortionary pricing. Lustig et al. (2009) develop this approach yet further and like us consider the impact of introducing long term bonds. In their model long bonds have the attraction of postponing and concentrating the increase in nominal interest rates that adverse fiscal shocks produce. The Lustig et al. model is one of incomplete markets, sticky prices, nominal bonds as well as long maturities. Their main focus is on extending the result of Chari et al (1991), about how inflation surprises influence nominal interest rates to achieve fiscal hedging in a model with long bonds.

3 Commitment and Independent Powers

We have so far followed the majority of the literature and assumed a Ramsey policy equilibrium with perfect commitment. Governments in Section 2 achieve lower current funding costs by promising lower future taxes but clearly this is a commitment governments would prefer to renege on. As well shall see, it is this promise to cut future tax rates in order to influence current funding costs that is at the heart of why solving optimal tax models under incomplete markets and long bonds is so computationally demanding. Optimal time consistent policy requires keeping track of all these promises over the last N periods and as the maturity of the bond increases so too does the state space.

3.1 Time Inconsistency - a Continuation Problem

We stated in equation (12) that a recursive formulation of the full commitment solution involves introducing N lags of b_N and λ as co-state variables. We now discuss the role of these variables in the solution and their link to interest rate twisting and compromises about future taxes. We show how these state variables appear in an equivalent continuation problem, justifying their role as co-state variables.

In this subsection we denote the Ramsey equilibrium as $\left\{c_t^R, b_{N,t}^R\right\}_{t=0}^{\infty}$. Assume the economy has been following the Ramsey equilibrium until some period $\bar{t} > 0$ and that, unexpectedly, the government can choose alternative policies in the future by maximizing

$$E_{\bar{t}} \sum_{t=0}^{\infty} \beta^t \left[u(c_{t+\bar{t}}) + v\left(x_{t+\bar{t}}\right)\right] \tag{26}$$

subject to equilibrium constraints (9), feasibility for $t = \bar{t}, \bar{t}+1, \ldots$ and given initial conditions $\left(g_{\bar{t}}, b_{N,\bar{t}-1}^R\right)$. In general, this solution would be different from the continuation of the Ramsey policy $\left\{c_t^R, b_{1,t}^R\right\}_{t=\bar{t}}^{\infty}$. This is the well known time inconsistency problem.

Time inconsistency arises for two reasons in this model: the government maximizing (26) will try to *i)* alter the cost of initial debt and, *ii)* it will "forget" promises that were previously made about future tax cuts (or tax increases) to promote interest rate twisting.

To discuss issue *i)* we consider the case $N = 1$. We claim that in this case, if the government in period $\bar{t}$ maximizes

$$E_{\bar{t}} \sum_{t=0}^{\infty} \beta^t \left[u(c_{t+\bar{t}}) + v\left(x_{t+\bar{t}}\right)\right] + \lambda_{\bar{t}-1}^R u_{c,\bar{t}} b_{1,\bar{t}-1}^R \tag{27}$$

subject to (9) for $t = \bar{t}, \bar{t}+1, ...$ the solution will be precisely $\left\{c_t^R, b_{1,t}^R\right\}_{t=\bar{t}}^{\infty}$. In other words, solving the continuation problem where the term $\lambda_{\bar{t}-1}^R u_{c,\bar{t}} b_{1,\bar{t}-1}^R$ is added to the utility function (26) delivers the Ramsey allocation from $\bar{t}$ onwards. The reader can convince herself of this statement by checking that the FOC derived from maximizing (27) coincide with the FOC from the Ramsey equilibrium for all periods $t \geq \bar{t}$. For a proof and a formal discussion see Marcet and Marimon (2014), Section 3.2 and Proposition 1.

The reason for this result (for $N = 1$) is that if $b_{1,-1}^g > 0$ the government would like to induce a high initial consumption, ie. there is a "bias for high c_0".[9] The reason is that this lowers interest paid on initial debt, an effect that can also be found in Lucas and Stokey (1983) under complete markets. If the government would maximize (26) at $\bar{t}$ there would be a "bias for high $c_{\bar{t}}$", leading the government to choose a $c_{\bar{t}} > c_{\bar{t}}^R$ in a time-inconsistent policy. The reason the term $\lambda_{\bar{t}-1}^R u_{c,\bar{t}} b_{N,\bar{t}-1}^R$ needs to be added to the objective function (27) in the continuation problem is that this term lowers the "total" marginal utility of $c_{\bar{t}}$ (since the marginal utility of $c_{\bar{t}}$ in (27) is $u_{c,\bar{t}} + \lambda_{\bar{t}-1}^R u_{cc,\bar{t}} b_{1,\bar{t}-1}^R < u_{c,\bar{t}}$). This avoids the "bias for high $c_{\bar{t}}$".

The second reason for time inconsistency is easier to see in the general case $N > 1$. We now claim that the equivalent continuation problem that delivers the Ramsey solution at $\bar{t}$ is to maximize

$$E_{\bar{t}} \left(\sum_{t=0}^{\infty} \beta^t \left[u(c_{t+\bar{t}}) + v\left(x_{t+\bar{t}}\right)\right] + \sum_{t=0}^{N-2} \beta^t \;\; \mathcal{D}_{t+\bar{t}}^R \;\; u_{c,t+\bar{t}} + \beta^{N-1} \;\; \lambda_{\bar{t}-1}^R \;\; u_{c,\bar{t}+N-1} b_{N,\bar{t}-1}^R \right) \tag{28}$$

The terms $\mathcal{D}_{t+\bar{t}}$ have been defined in (14) and they modify the weight that consumptions c_t receive for $t = \bar{t}, \ldots, \bar{t}+N-2$. These consumptions need to be reweighted because, as we have explained in section 2.1.1, optimal Ramsey policy involves promises about consumption N periods ahead in order to twist current interest rates. Such promises were made in periods $t = \bar{t}, \bar{t}-1, ..., \bar{t}-N+1$

[9] This can be seen in the optimality conditions (10) because, given that $\lambda_{-1} = 0$, the term $\left(\lambda_{t-1} - \lambda_t\right) b_{1,t-1}$ is definitely negative for $t = 0$, while the same term can be of either sign for $t > 0$ since $E_{t-1}\left[\left(\lambda_{t-1} - \lambda_t\right) b_{1,t-1}\right]$ is approximately zero due to the martingale property of multipliers.

that involve consumptions for $t = \bar{t}, \ldots, \bar{t} + N - 1$. The terms $\mathcal{D}_{t+\bar{t}}$ that appear in (28) are needed to guarantee that these promises are satisfied. The last term in (28) appears because the "bias for high c_0" becomes a "bias for high c_{N-1}" in the presence of $N-$period bonds, as highlighted by the example in section 2.2.1.

Notice that the terms added to agents' utility in (28) involve N lags of λ, this is why these multipliers are part of the state vector since they influence the objective function of the continuation problem along with N lags of b_N.

This discussion also clarifies why time inconsistency arises in models of incomplete markets: in a model of long bonds we see how the effect from interest twisting is separate from the "initial consumption bias" issue. These two effects are confounded in one period for short bonds, when $N = 1$.

The above discussion also highlights why the lagrangean approach of Marcet and Marimon (2014) is easier to apply to models of optimal policy over the promised utility approach. The latter would require to compute the feasible set of N promised utilities (or in this case promised *marginal* utilities $u_{c,t}$) so as to promise policies that in the future can actually be equilibria. Computation of the feasible set of marginal utilities can be highly involved. The lagrangean approach sidesteps the computation of this set, the lagrange multipliers λ do not need a restriction of that type because the objective functions (27) and (28) give a well defined maximization problem for any value of λ's.

3.2 Independent Powers

The previous discussion shows that interest rate twisting arises because of the close connection between current interest rates and future tax policy in our model. In this section we consider a different institutional set up, one of independent powers, such that governments cannot commit to influence future tax rates in order to affect current funding costs[10].

More specifically, we relax the assumption of perfect coordination and assume the presence of a monetary policy authority[11] that fixes interest rates in every period. The fiscal/debt management authority now takes interest rates as given and implements optimal policy given these interest rates, knowing the relation between taxes and allocations given by (5) and feasibility. We examine an equilibrium where the two policy makers play a dynamic Markov Nash equilibrium with respect to the strategy of the other policy power and they both play Stackelberg leaders with respect to the consumer. More precisely, the fiscal authority chooses taxes and debt given a sequence for interest rates, the monetary authority simply chooses interest rates that clear the market and the fiscal authority maximizes the utility of agents. This assumption sidesteps the issues of commitment, now there is no room for interest rate twisting on the part of the fiscal authority since this agent takes interest rates as given.

[10]Debortoli, Nunes and Yared (2015) examine the case when governments cannot commit to future tax policies and focus on Markov Perfect Competitive Equilibrium rather than our institutional separation of powers. They also use the complete market solution method of Angeletos (2002) to solve for the optimal portfolio model rather than numerical state space based approaches.

[11]In practice there are three relevant agencies - a fiscal authority, a debt management office and a central bank. In our simple model we can think of interest rates as either being set independently by the central bank or the debt management office taking interest rates as given by the market and operating independently of the fiscal authority. What we propose in this section is not intended as an accurate description of how interest rates are set in the economy but merely to show the implications of independence between the two authorities.

It is easy to think of models where even if the monetary authority is independent it cannot deviate too much from equilibrium interest rates. Therefore we take a limit case and assume that the monetary authority simply sets interest rates in equilibrium as:

$$\begin{aligned} p_{N,t} &= \frac{\beta^N E_t\left(u_{c,t+N}\right)}{u_{c,t}} \\ p_{N-1,t} &= \frac{\beta^{N-1} E_t\left(u_{c,t+N-1}\right)}{u_{c,t}} \end{aligned} \tag{29}$$

given agent's consumption. Formally we use the following

Definition An equilibrium under independent powers (IP) is a sequence of bond prices $\{p_{N,t}, p_{N-1,t}\}$ each contingent on g^t, such that if the fiscal authority solves

$$\max_{\left\{c_t, b_{N,t}\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{u\left(c_t\right) + v\left(x_t\right)\right\} \tag{30}$$

$$\text{s.t.} \qquad p_{N-1,t} b_{N,t-1} = \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)\left(c_t + g_t\right) - g_t + p_{N,t} b_{N,t},$$

(1) and (4) taking bond prices as given then (29) holds.

We look for equilibria where bond prices are given by an interest rate policy function $\mathcal{R} : R^2 \to R^2$ that satisfies

$$\left(p_{N,t}, p_{N-1,t}\right) = \mathcal{R}(g_t, b_{N,t-1}), \tag{31}$$

although the relation $\mathcal{R}$ is ignored by the authority solving (30).

An advantage of this model is that within equilibria of the form (31) there is no longer any reason for longer lags to enter the state vector, as past Lagrange multipliers do not play a role. From the point of view of the fiscal authority the problem now is a standard dynamic programming problem with the vector of state variables $(b_{N,t-1}, g_t)$.

Multiplying both sides of the budget constraint by $u_{c,t}$ the Lagrangian of (30) becomes

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ & u\left(c_t\right) + v\left(x_t\right) + \lambda_t \left[S_t + u_{c,t}\left(p_{N,t} b_{N,t} - p_{N-1,t} b_{N,t-1}\right)\right] \\ & + \nu_{1,t}\left(\overline{M} - \beta^N b_{N,t}\right) + \nu_{2,t}\left(\beta^N b_{N,t} - \underline{M}\right) \} . \end{aligned} \tag{32}$$

The first order condition with respect to consumption combined with the budget constraint gives

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t}\left(c_t + g_t\right) - v_{x,t}\right) - u_{cc,t} \lambda_t S_t / u_{c,t} = 0. \tag{33}$$

In addition, the FOC with respect to bonds combined with (29) gives

$$\lambda_t E_t\left(u_{c,t+N}\right) = E_t\left(\lambda_{t+1} u_{c,t+N}\right) + \nu_{2,t} - \nu_{1,t}. \tag{34}$$

An IP equilibrium can be computed using these two equations along with the government budget constraint, debt limits with their slackness conditions, feasibility and the fact that the only state variables are $(b_{N,t-1}, g_t)$.

Notice that (34) takes the same form as the FOC under full commitment (11). Therefore λ_t is again a risk-adjusted martingale off corners. Obviously, the value of λ will be different than in the Ramsey equilibrium since consumption follows a different process under IP.

4 Stochastic Simulations

We now turn to a model where g_t is stochastic in all periods. We assume a utility function:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2}.$$

We choose $\beta = 0.98$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $A = 100$. We set η such that if the government's deficit equals zero in the non stochastic steady state agents work a fraction of leisure equal to 30% of their time endowment. For the stochastic shock g we assume the following truncated AR(1) process:

$$g_t = \begin{cases} \overline{g} & \text{if } (1-\rho)\, g^* + \rho g_{t-1} + \varepsilon_t > \overline{g} \\ \underline{g} & \text{if } (1-\rho)\, g^* + \rho g_{t-1} + \varepsilon_t < \underline{g} \\ (1-\rho)\, g^* + \rho g_{t-1} + \varepsilon_t & \text{otherwise} \end{cases}.$$

We assume $\varepsilon_t \sim N(0, 1.44)^2$, $g^* = 25$, with an upper bound $\overline{g}$ equal to 35% and a lower bound $\underline{g} = 15\%$ of average GDP and $\rho = 0.95$. $\overline{M}$ is set equal to 80% of average GDP and $\underline{M} = -\overline{M}$.

4.1 Solving the Model with "Condensed PEA"

We solve the model applying the Parameterized Expectations Algorithm (hereafter PEA) of den Haan and Marcet (1990) to approximate numerically the terms that appear in the equilibrium conditions $E_t\left(u_{c,t+N}\right)$, $E_t\left(u_{c,t+N-1}\right)$ and $E_t\left(u_{c,t+N}\lambda_{t+1}\right)$ as functions of the state variables. As highlighted before, in the model of Section 2 the dimension of the state vector is $2N+1$ which even if we only consider bonds of 10 year maturities produces a state space of 21.

Faraglia, Marcet, Oikonomou and Scott (2014 a and b) suggest that in order to make the computation of models with large N manageable it is important to reduce the number of states which enter autonomously in the approximating polynomials. Using a refinement of the PEA called the "Condensed PEA", their approach is to partition the state space into variables that are of primary importance for the solution and variables of secondary importance. The latter are introduced in the approximating functions as successive linear combinations. We apply this methodology to solve the commitment model and refer the reader to Faraglia et al. (2014 a and b) for an extensive discussion. The independent powers model with its state vector of only two variables $(g_t,\, b_{t-1}^N)$ is solved applying the standard PEA.

To approximate the optimal policy accurately we make sure that we visit all possible realizations of the state vector with our simulations. This is more of an issue in our model since government debt is very persistent and therefore it may be expected that different realizations of spending, or different initial conditions of debt, may make the debt and tax series follow considerably different paths. Our approximation to the parameterized expectation is based on 14000 samples each of 200 observations and with initial conditions for bonds uniformly distributed in the interval $[\underline{M}, \overline{M}]$. When we later report our simulation results we change our sample and describe the model's performance over different horizons for given initial conditions.

4.2 Interest Rate Twisting

Figures 1 and 2 display the impulse response functions of key variables to an unexpected positive shock in g_t. The vertical axis is in units of each of the variables and expresses deviations from the value that would occur for the given initial condition if $g_t = g^*$. Each subplot shows two lines: the solid line represents the solution under full commitment of Section 2, the dashed line represents the case of the "independent powers" model of Section 3.2. Both figures are for a maturity $N = 10$.

[Figures 1 and 2 About Here]

Figure 1 presents the result when the government has zero inherited debt, $b_{N,-1} = 0$. The differences between the two models should highlight the effect of the government keeping past promises summarised by the variable $\mathcal{D}_t$. In this case there is no effect even under full commitment since $\mathcal{D}_N = 0$. As the Figure shows the rise in spending leads to an initially smaller but more persistent increase in taxes in the case of full commitment than under independent powers. However the effect is moderate leading to only small differences. The two models are similar.

Figure 2 shows the results assuming a positive initial debt equal to $b_{N,-1} = 0.5\ y^*/\beta^N$ where y^* is steady state output. There is a blip in taxes at the time of maturity of the outstanding bonds $N = 10$, reflecting the promise to cut taxes with the aim to twist interest rates as discussed in Section 2.2. Interest rate twisting, and the blip in period $t + N - 1$, occurs each period g_t is high if the government is in debt. The size of the promised tax cut at $t + N - 1$ depends on how big are relative past shocks, $(\lambda_{t-1} - \lambda_t)$, and debt, $b_{N,t-1}$. Besides the stronger persistence, the tax rate with commitment shows clearly the effect to reduce the tax rate and increase consumption $N - 1$ periods after the shock. These anticipated changes affect also the deficit and the market value of government debt as illustrated in the bottom panels.

Obviously, the IP model does not show the blip in $N - 1$ periods although other than that the responses are similar in the two models. The only notable difference is that in periods other than $N - 1$ the response of taxes is smoother under full commitment, reflecting the fact that interest rate twisting has the beneficial effect of smoothing taxes in periods other than $N - 1$.

4.3 The Impact of Maturity

To further illustrate the link between the maturity of debt and interest rate twisting, we plot in Figure 3, the response of taxes[12] to the shock under four different maturity structures, $N = \{5, 10, 15, 20\}$ The top left panel shows the case of commitment and zero initial debt, the top right high debt with commitment. The bottom panels illustrate the response of the tax schedule in the independent powers model.

[Figure 3 About Here]

[12]For the case $b_{N,-1} = 0.5y^*/\beta^N$.

Consistent with the previous results all tax responses in the top right panel show the interest rate twisting effect. Given our previous discussion it is clear why the blip in taxes keeps moving to the right as we increase the maturity. In the case of zero debt, as well as in the case of independent powers, the maturity structure shows little effect on optimal taxes.

4.4 Moments

We now evaluate the model properties reporting the first and second moments of some key model variables. In the first four rows of Table 1 and 2 we show the means of consumption, taxes, deficit and market value of debt for $N = 5$, 10, 15, 20. In the last four rows we report the standard deviations of these variables in our simulations. The means and standard deviations are evaluated over three different horizons: 40 periods (columns 1-4), 200 periods (columns 5-8) and 4500 periods[13] (columns 9-12). These three cases enable us to clearly identify the influence of initial conditions on policy outcomes.

Table 1 reports the result for the model with commitment. With the exception of debt and deficit all the moments differ only to the second or third decimal place across maturities. However, with the government only issuing one type of bond in each case, smoothing taxes is mainly achieved by using debt as a buffer stock so that the fluctuations of the model variables are driven mostly by the strong low frequency fluctuations of debt leaving only a relatively minor impact of interest rate twisting on total variance.

The main exception are the levels of debt and deficit: the government in the long run holds assets, but average asset holdings are lower for higher maturities. As is well known, in models of optimal policy with incomplete markets, there is a force pushing the government to *accumulate* long bonds in the long run. More precisely, extending the results in Aiyagari et al (2002) Section III one can easily prove that in the case of linear utility ($u(c) = c$) the government would purchase a very large amount of private long bonds in the long run, enough to abolish taxes. This accounts for the negative means for debt shown in Table 1 and for the significant differences in the means of the market value of debt which occur at longer horizons in the simulations.[14] On the other hand, as argued in Angeletos (2002), Buera and Nicolini (2004) and Nosbusch (2008), if the term premium is negatively correlated with deficits (as it is in our model) it is optimal for the government to *issue* long bonds, as this provides fiscal insurance. Hence the government is aware that accumulating a very large amount of privately issued long bonds increases the volatility of taxes. This force accounts for the *lower* asset accumulation with *longer* maturities shown in Table 1.

To identify the effect of commitment we report the same moments for the "independent powers" models in Table 2. Comparing Table 1 and Table 2, it is evident that across all horizons and across all maturity structures, the effect of the interest rate twisting channel is small.

[Tables 1 and 2 About Here.]

To conclude, under the standard assumptions on long bonds, namely that they pay zero coupons and are purchased one period after issuance, the interest rate twisting policy channel is apparent but it

[13] To get rid of the influence of initial conditions we dropped the first 500 observations from each sample in columns 8-12.

[14] As the table shows the average market value becomes considerably more negative in the long run.

does not substantially influence unconditional first and second moments. The model of "independent powers" may be a good model to have in the toolkit as it retains many of the interesting features of the Ramsey models, it has nearly the same moments, it avoids the technicalities arising from the very large state vector and it avoids discussion on the role to commitment at very long horizons.

5 No-Buyback and Rollover Cycles

In this section we extend our model by assuming the government never repurchases previously issued bonds, so that N-period government bonds are redeemed by the government N periods after issuance. This is an extreme assumption: the US government has sometimes repurchased its own bonds, but as shown in Faraglia et al (2014b) it only does so close to redemption date. Therefore, the no-buyback assumption of this section is much closer to actual US practice than the buyback assumption that is standard in the literature and Sections 2 to 4. As with interest rate twisting the implications of no-buyback are absent with short bonds, since there is no room for the government to buy debt back before it matures if $N = 1$.

Under no-buyback the budget constraint of the government becomes

$$b_{N,t-N} = \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)(c_t + g_t) - g_t + p_{N,t}b_{N,t}. \tag{35}$$

Now a bond issued pays a given amount in N periods, while in Section 2 it paid an uncertain amount $p_{N-1,t+1}$ next period. Under incomplete markets this is not without loss of generality as the timing of cash flows matter and so assuming no early buy back will lead to different outcomes. Also, this budget constraint shows how long bonds only connect every $N-$th period: a large deficit in t drives $b_{N,t}$ upwards due to a standard buffer stock effect, the cost of higher future payments is only borne in periods $t + N$, $t + 2N$, ..., thus taxes in these periods are higher than in the interim periods, hence taxes vary across periods. With one-period bonds the burden of higher debt servicing would be spread out across all future t's and this would help to smooth taxes.

Imposing the bond limits (4) in the current setup for a given value for $\underline{M}, \overline{M}$ would result in a larger total debt, since the government now holds on to N lags of previously issued bonds. Therefore we modify the bond limits to

$$\sum_{j=1}^{N} \beta^j b_t^i \in \left[\underline{M},\ \overline{M}\right] \tag{36}$$

since $\underline{M}, \overline{M}$ now give the same upper bound on the total value of debt as in the previous sections when past bonds are valued at steady state prices $p_t^i = \beta^i$ for maturity i.

Building a Lagrangian in an analogous way as we did in Section 2 gives that, off corners, the first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are[15]

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}\right) + u_{cc,t}\left(\lambda_{t-N} - \lambda_t\right) b_{N,t-N} = 0 \tag{37}$$

$$E_t\left(u_{c,t+N}\lambda_{t+N}\right) = \lambda_t E_t\left(u_{c,t+N}\right). \tag{38}$$

[15]See Faraglia, Marcet, Oikonomou and Scott (2014b) Section 6, for details on the Lagrangean and FOC.

5.1 Impossibility of completing markets with a long bond

For a striking example of the impact of no early buyback, suppose as in the example of Section 2.2.1 that there is no uncertainty and $g_t = \overline{g}$ for all t. To simplify even further, assume maturity $N = 2$ (although our results are valid for any N). Moreover assume the government inherits some non-zero debt $b_{2,-1}$, $b_{2,-2}$ and that the (finite) bond limits $\underline{M}$, $\overline{M}$ can be chosen arbitrarily large.

Since we have no uncertainty and bond limits can be arbitrarily large, it may seem at first sight that one bond completes the markets, so that all equilibrium constraints are summarized in a single implementability constraint setting

$$\sum_{t=0}^{\infty} \beta^t \frac{S_t}{u_{c,0}} = b_{2,-1}\beta\frac{u_{c,1}}{u_{c,0}} + b_{2,-2}. \tag{39}$$

But it turns out that this constraint is not sufficient for an equilibrium: a long bond does not complete the markets, even under certainty.

To see this consider the optimal allocation when (39) is the only implementability constraint. It is clear that optimal taxes would be constant for $t \geq 2$ so that

$$S_t = \overline{S},\ u_{c,t} = \overline{u}_c \quad t = 2, 3, \ldots \tag{40}$$

The analog of equation (39) at period t gives that the bonds that implement this allocation satisfy

$$b_{2,t} = \frac{\overline{S}/\overline{u}_c}{(1-\beta)\beta} - \frac{1}{\beta} b_{2,t-1} \qquad t = 1, 2, \ldots \tag{41}$$

Since $\frac{1}{\beta} > 1$ this is an explosive difference equation in $b_{2,t}$. Here $b_{2,t}$ would alternate in sign and go to infinity in absolute value, thereby violating the bond limits (36) for any finite $\underline{M}$, $\overline{M}$.[16]

This shows that (39) can not be the only implementability constraint, because there are no bond allocations that implement the optimal consumption allocation under this constraint.

What is going on? The problem is that the standard present value condition (39) is derived under the assumption that the market value of debt $b_{2,t-1}\beta + b_{2,t-2}$ remains bounded and, indeed, it does in this example. But bounded market value of debt goes along with bond limits that explode in absolute value, and this is ruled out by the bond limits (36). It is reasonable to impose bond limits and not only limits to total value: if $b_{2,t-1}$ and $b_{2,t-2}$ are eventually huge in absolute value and of opposite signs (as determined by (41)), the government would hold very large amounts of private debt and it would risk very high losses from a private default.

To summarize, with bond limits (39) is not a sufficient implementability condition. In fact, by forward substitution in (35) one can see that under certainty a set of sufficient implementability

[16] Note that the values of initial conditions $b_{2,-1}$, $b_{2,-2}$ are independent on the constant $\frac{\overline{S}/\overline{u}_c}{1-\beta}$ for a given initial value of debt, so that no end condition holds to guarantee that the difference equation (41) is generically stable. There is one configuration of $b_{2,-1}, b_{2,-2}$ that does imply stability for a given wealth level $b_{2,-1}\beta\frac{u_{c,1}}{u_{c,0}} + b_{2,-2}$, but this would only happen by coincidence, almost all combinations of $b_{2,-1}, b_{2,-2}$ that give rise to the same wealth imply $|b_{2,t}| \to \infty$.

conditions is given by the following *two* conditions

$$b_{2,-2} = \sum_{t=0}^{\infty} \beta^{2t} \frac{S_{2t}}{u_{c,0}} \tag{42}$$

$$b_{2,-1} = \sum_{t=0}^{\infty} \beta^{2t} \frac{S_{2t+1}}{u_{c,1}}. \tag{43}$$

These two conditions imply (39). But the converse is not true, many allocations satisfy (39) but violate (42)-(43), including the optimal solution with (39). That is the optimal policy under (39) implies that bond issuance goes to infinity and the bond limits are violated.

Intuitively: under no-buyback and $N = 2$ there is no way to transfer income between odd and even periods. High income in, say $t = 1$, can be transferred to, say $t = 7$, but not to $t = 2$. Therefore, for most initial conditions $b_{2,-2}, b_{2,-1}$ even and odd periods will have a different primary surplus so that (40) can not hold. This proves the following:

Optimal policy in the example of this section is

$$\begin{aligned} \tau_t &= \tau^o \text{ for all } t \text{ odd} \\ \tau_t &= \tau^e \text{ for all } t \text{ even} \end{aligned} \tag{44}$$

where $\tau^o \neq \tau^e$ *generically.*

Obviously, this implies that for the optimal policy there is an $N-$period cycle in bonds in this example, hence

$$b_{2,t} = b_{2,t-2} \text{ for all } t \geq 3. \tag{45}$$

In this setup, long bonds impart tax variability, the opposite of fiscal insurance. Tax smoothing takes place within odd periods and within even periods but not across all periods. Only when $b_{2,-2} = b_{2,-1}$ can we implement the complete markets allocation in this example.

5.2 Some analytic Results

Maintaining certainty and for arbitrary N note that from the standard first order conditions it holds that[17]

$$\lambda_t = \lambda_{t+N} \text{ for all } t, \tag{46}$$

entailing that the multiplier λ_t follows an N cycle, since λ's repeat every N periods but generally $\lambda_t \neq \lambda_{t+1} \neq \ldots \neq \lambda_{t+N-1}$. Taxes and consumption inherit this N cycle property. Furthermore, interest rate twisting now causes taxes in the first N periods to respond to a shock differently depending on current debt. This is because now the government in period t holds bonds issued at $t-1, \ldots, t-N$ and aims at twisting the interest rates of all these bonds to lower funding costs. We state this with the following result:

Result 2. *Assume no-buyback, an arbitrary non-random* $\{g_t\}$, *and the utility function in (21).*

[17] This follows from (38) and an argument parallel to the one leading to equation (17).

Ramsey equilibrium is that there are cycles of order N in taxes for $t \geq N$. More precisely

$$\tau_t = \tau_{t+iN} \qquad t = N, \ldots, 2N-1 \text{ for all } i = 1, 2, \ldots \tag{47}$$

Assume further $b_{N,-i} > 0$ for $i = 1, .., N$ then

$$\tau_{i+N} > \tau_i \quad i = 0, \ldots, N-1. \tag{48}$$

Proof.

We give details for $N = 2$, it is trivial to extend the proof to arbitrary N.

Equation (46) implies $\lambda_t = \lambda_{t-2}$ for all $t \geq 2$. Plugging this into the first order condition of the problem we have

$$\begin{aligned} u_{c,t} - v_{x,t} + \lambda_0 \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + \overline{g} \right) - v_{x,t} \right) &= 0 \text{ for all } t \geq 2, \quad t \text{ even} \\ u_{c,t} - v_{x,t} + \lambda_1 \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + \overline{g} \right) - v_{x,t} \right) &= 0 \text{ for all } t \geq 3, \quad t \text{ odd.} \end{aligned} \tag{49}$$

A standard derivation gives that for utility function (21) we have

$$\begin{aligned} \tau_t &= \tau_2 \text{ for all } t > 2, \quad t \text{ even} \\ \tau_t &= \tau_3 \text{ for all } t > 3, \quad t \text{ odd.} \end{aligned} \tag{50}$$

This proves (47) for $N = 2$.

Finally we show (48). For periods $t = 0, 1$ we have

$$\begin{aligned} u_{c,0} - v_{x,0} + \lambda_0 \left(u_{cc,0} c_0 + u_{c,0} + v_{xx,0} \left(c_0 + \overline{g} \right) - v_{x,0} \right) - u_{cc,0}\, \lambda_0\, b_{2,-2} &= 0 \\ u_{c,1} - v_{x,1} + \lambda_1 \left(u_{cc,1} c_1 + u_{c,1} + v_{xx,1} \left(c_1 + \overline{g} \right) - v_{x,1} \right) - u_{cc,1}\, \lambda_1\, b_{2,-1} &= 0 \end{aligned} \tag{51}$$

The only difference between equations (51) and (49) is the presence of two extra terms that are function of the initial condition of debt: $u_{cc,0}\ \lambda_0\ b_{2,-2}$ and $u_{cc,1}\ \lambda_1\ b_{2,-1}$. Since we have assumed that $b_{2,-2}, b_{2,-1} > 0$ these terms are clearly negative, implying that

$$\begin{aligned} \tau_2 &> \tau_0 \\ \tau_3 &> \tau_1. \ \blacksquare \end{aligned}$$

This extends the result (44) to the case when g is time-varying but restricting the utility function to (21). As discussed in Section 5.1, a long bond under no-buyback and bond limits does not complete the markets, all dates $t + iN$ are now isolated for different $t = 0, \ldots, N-1$. Therefore the $N-$period cycle arises because of the budget constraint, not because of the way we model policy or because of interest-rate twisting. To show more clearly that (47) depends on the budget constraint and not on commitment, in an appendix we consider independent powers under no-buyback and find N-period cycles emerge in that model as well. Notice that (45) does not generalize to the case of arbitrary $\{g_t\}$.

The inequality (48) shows that the government commits to twisting interest rates for *all* first N periods. This is in contrast with (23) in Result 1 without buyback, where interest rates were only twisted around the $N-$th period. Under no-buyback the government holds bonds issued in the last $N-1$ periods so it commits to lowering taxes for the next $N-1$ periods in order to cut the cost of all bonds outstanding.

The following is a special case of Result 2 when $\{g_t\}$ follows the certainty analog of the AR(1) process we use later in our simulations. This result gives intuition for the evolution of taxes we find in Figure 4 below.

Result 3. *Consider the assumptions of Result 2 except that, to isolate from interest rate twisting, we assume $b_{N,-i}=0$ for $i=1,\ldots,N-1$.*

Assume in addition, that $g_t=(1-\rho)\,g^*+\rho g_{t-1}$ for $g_0>g^*>0$ and $\rho\in(0,1)$.

Then

$$\tau_t>\tau_{t+1}\qquad t=0,\ldots,N-1. \tag{52}$$

Proof.

As argued in section 5.1 the budget constraints only link the periods of cycle N, so that the set of sufficient implementability conditions can be written as

$$0=\sum_{i=0}^{\infty}\beta^{Ni}\frac{u_{c,t+Ni}}{u_{c,t}}\left[g_{t+Ni}-\tau_{t+Ni}\left(A-x_{t+Ni}\right)\right]\text{ for }t=0,\ldots,N-1. \tag{53}$$

For the g process assumed here g_t is decreasing geometrically. Therefore it is clear that the discounted sum of expenditures decreases as t grows from 0 to $N-1$, formally

$$\sum_{i=0}^{\infty}\beta^{Ni}\frac{u_{c,t+Ni}}{u_{c,t}}g_{t+Ni}>\sum_{i=0}^{\infty}\beta^{Ni}\frac{u_{c,t+1+Ni}}{u_{c,t+1}}g_{t+1+Ni}\qquad t=0,\ldots,N-1.$$

Therefore, if (53) must hold the discounted sum of tax revenues also has to go down as t grows from 0 to $N-1$. Since τ has to be in the increasing part of the Laffer curve in order to be an optimal tax, it means that taxes go down as t grows from 0 to $N-1$.$\blacksquare$

This says that for the deterministic analog of the AR(1) process used in the simulations, taxes will go down within each N-period cycle. Since (47) still applies, taxes initially decrease for N periods, there is a jump at $t=N$ to set $\tau_N=\tau_0$, from then on taxes decrease again until $t=2N-1$, there is a jump at $t=2N$ to set $\tau_{2N}=\tau_0$ and so on. Therefore, this is very similar to the dashed line we find in Figure 4 below.

5.3 Interest Rate Twisting under no-buyback

We now argue that, as in Section 3, interest rate twisting also takes place under no-buyback in all periods in response to shocks and that the role of the λ's is to enforce promises involved in the commitment to change future taxes. But this twisting takes a very different form. Now the presence of an adverse shock causes all taxes during the next N-period cycle to be slightly different than in previous cycles, causing an analog response as in (48) but for all periods. This is because under no-buyback the government owes bonds of maturities $1,\ldots,N-1$, since long bonds issued $N-1$ periods ago have not yet been redeemed. Therefore the government promises cuts in taxes in order to affect consumption during this first cycle since each of them individually influences the value of currently held debt.

Looking at the FOC of the no-buyback problem (37) and (38), there are only two differences with respect to the buyback case in (10) and (11), now we have:

1. λ_{t+N} in (38) in place of λ_{t+1} in (11) and

2. λ_t in (37) in place of λ_{t-N+1} in (11).

The first difference implies that the martingale property of λ in Section 2 (see our discussion after (13)) now only holds every $N-$th period. This generates the $N-$period cycles in taxes that we have already discussed. The second difference produces a more subtle effect. Notice that the term that induces interest rate twisting is now $(\lambda_{t-N} - \lambda_t)\, b_{N,t-N}$ instead of $\mathcal{D}_t$ as defined in (14). The difference $(\lambda_{t-N} - \lambda_t)$ depends on *all* the shocks that have happened between $t - N$ and t, while in Section 2 we had $(\lambda_{t-N} - \lambda_{t-N+1})$ so that only the shock occurring at $t - N + 1$ mattered. Due to (38) we should have $\lambda_{t-N} \simeq \lambda_t$ if all shocks are close to the mean between $t - N$ and t, but if negative (positive) shocks to g happen between $t - N$ and t the realized values will be $\lambda_{t-N} < \lambda_t$ ($>$) and the interest-rate twisting term will induce a lower (higher) consumption at t. This implies that a shock in period t induces interest rate twisting for all taxes in periods $t, \ldots, t + N - 1$. In this sense the effect of a shock to g_t on interest rates twisting is spread out over periods $t + 1, ..., t + N$. This reflects a stochastic interpretation of the analytic result (48) that occurred in Result 2.

5.4 Simulations

5.4.1 Business Cycle Fluctuations

Consider again our simulations of Section 4 but now amend the model for the case of no buy back. The calibration is the same as in the previous section, including initial value of debt and the total value for bond limits $\underline{M}$, $\overline{M}$. We start with the case when initial conditions are symmetric, $b_{N,-j} = b_{N,-i}$ for all $i, j = 1, ..., N$. In section 5.4.2 we consider an alternative where initial conditions are asymmetric.

Figure 4 shows how taxes respond to an adverse government expenditure shock when the government has initial debt equal to half of GDP. This Figure compares the case of buy back at the end of each period (as in Section 2) and no-buyback zero coupons as in this Section. We see that the behavior described by Results 2 and 3 arises: there are $N-$period cycles and taxes go down within each cycle. Under buy back we saw clearly the interest rate twisting effect but under no-buyback things are significantly different. The interest rate twisting effect is spread out across each of the periods and it can be barely seen, however the $N-$period cycles due to cash flow requirements when debt is rolled over are obvious.

[Tables 3 and 4 About Here]

Table 3 shows second moments of several variables found with repeated simulations. Comparing with Table 1 shows that under no buy back the deficit and market value of debt are larger on average and that taxes, deficit and consumption become more volatile for most maturities and most horizons although the increase in volatility due to no-buyback is quantitatively minor.

The reason for the minor change is the following: from our discussion around the end of Section 5.1 it is clear that the volatility across different $N-$period cycles is due to differences in the initial conditions $b_{N,-i}$ across $i = 1, ..., N$. For the simulations summarized in Table 3, given our calibration, surprises in g in any given period are relatively small. Furthermore, since debt is very persistent under incomplete markets (as emphasized, for example, in Aiyagari et al. (2002) and Marcet and Scott (2009)), there are no large differences in state variables $b_{N,t-i}$ across $i = 1, ..., N$. Therefore we expect that the different taxes across $N-$period cycles are small.

The next section describes a relevant situation where tax volatility does occur under no-buyback due to asymmetric initial conditions.

5.4.2 Tax volatility after a war

Consider an economy that has experienced large shocks to its deficit in the last few years and the impact this would have, if only long bonds under no-buyback can be issued. As a reference, consider the huge US deficits between 1942-45 of roughly 25%[18], such that the initial period $t = 0$ is represented by year 1946, assume zero coupons and ten year bonds ($N = 10$). Under incomplete markets high deficits translate into high bond issuance, this would justify calibrating initial conditions as $b_{N,-j} = 25\%GDP$ for $j = 1, ..., 4$. Consider, for simplicity, a situation where the government had zero debt before 1942 so that $b_{N,-j} = 0$ for $j = 5, ..., 10$. The upper bond limit corresponds to a maximum value of debt/GDP ratio of 100% so that $\overline{M} = 100\%GDP$. We assume the government can not buy private bonds, i.e. $\underline{M} = 0$. All other parameters remain as in the previous calibration. We compare this with a buyback model calibrated in an analogous way.[19]

Table 5 shows tax volatility after exiting the war under buyback and no-buyback. As we explained, buyback is closer to a one-period bond in terms of maturity and, therefore, it allows for tax smoothing after war. But as suggested by the results in section 5.1 the high levels of bond issuance for $j = 1, ..., 4$ reverberate into very high interest rate payments in ten-year cycles and, therefore, high taxes every 10 years. Tax volatility is four times higher under no-buyback in the first 20 years due to this cycle. Tax volatility goes down as time goes by, since bonds revert to a situation where state variables are symmetric in the long run.

[Table 5 About Here]

These results are robust to many changes. For example, if we loosen the lower bound of debt to $\underline{M} = -\overline{M}$ the standard deviation of taxes is still about four times larger: 0.024 for buyback and 0.095 for no-buyback at 20-year horizon. Even if we double the bounds $\overline{M}$ for the no-buyback case, therefore giving a much better chance for tax smoothing under no-buyback, standard deviation of taxes is still very large: 0.061, 0.052, 0.048 at the horizons 20, 40, 60.

[18]Respectively 14.2, 30.3, 22.7 and 21.5% from 1942 to 1945 of GDP, according to http://www.econdataus.com.

[19]More precisely, we set up initial conditions so that the value of initial debt (at steady state bond prices) is exactly the same both under buyback and no-buyback. More precisely, in the model with buyback we assume that $\beta^{N-1} b_{N,-1} = y_{ss}$ and $b_{-j} = 0$ for $j = 2, 10$, where y^{ss} is output with a balanced budget and $g_t = E(g_t)$. In the model with no buyback we assume $\beta^{j-1} b_{N,-j} = 0.25 y_{ss}$ for $j = 1, 4$ so that total value of initial debt is $\sum_{j=1}^{10} \beta^j b_{N,-j} = y_{ss}$.

5.4.3 The Benefits of Shorter Duration

The results here show the importance of modelling explicitly when long bonds are repurchased. This has implications for debt management and it suggests that, given no-buyback, shorter average bond duration may be helpful for tax smoothing.

In particular, Faraglia, Marcet, Oikonomou and Scott (2014b) show that short bonds can play a role at business cycle frequencies. The results in the previous Section 5.4.2 suggest that more flexibility in maturity plays an important role after a war. Although we do not develop this issue here, we provide the following intriguing fact: the US government did issue large amounts of callable bonds precisely during WWII, but callable bonds progressively disappeared and they were no longer issued after 1982.[20] Section 6 shows that tax volatility decreases when long bonds pay coupons and, therefore, their duration is shorter.

6 Coupon Bearing Bonds

A final issue we consider in modelling long bonds is the effect of introducing coupon payments. In practice long bonds invariably pay a coupon at fixed regular intervals with the coupon fixed for the duration of the bond (see FMOS (2014b) for documentary evidence). In the case of one period bonds coupons are unimportant - if coupons are paid at the end when the bond is redeemed all interest payments are paid at the maturity date and the duration of a bond is the same as its maturity. If we assume buyback then the impact of coupons is uninteresting as cash flows are unaffected. But if $N > 1$ and there is no early buyback then coupons make a substantive difference as duration will no longer equal maturity.

In terms of the rollover cycles of the previous section by spreading interest payments over the life of the bond and so reducing duration paying coupons should smooth taxes and reduce the N period cycles. However the volatility of the price of a bond is a direct function of its duration so whilst coupon payments will reduce the magnitude of N cycles they will also reduce the ability of long bonds to provide the fiscal insurance that the optimal debt management literature has to date emphasised. Given this it is worth investigating how the introduction of coupons affects the interest rate twisting, N-period cycles and tax volatility we have identified above in a model with no-buyback.

Let κ_t be the coupon payment of a bond issued at t, this payment is constant from t to $t+N-1$. In order to denote that non-zero coupon bonds have a different equilibrium price than zero coupon bonds, let q_t^N be the price of such a bond. In equilibrium

$$q_t^N = \kappa_t \sum_{i=1}^{N-1} \beta^i E_t \left(\frac{u_{c,t+i}}{u_{c,t}} \right) + \beta^N E_t \left(\frac{u_{c,t+N}}{u_{c,t}} \right) \tag{54}$$

i.e. q_t^N is the sum of prices of zero coupon bonds of maturity $j < N$ ($p_t^j = \beta^j E_t \left(\frac{u_{c,t+i}}{u_{c,t}} \right)$) weighted by the coupon payments promised, plus an N period zero-coupon bond that pays one unit of consumption at maturity (a normalization). We call this a "fixed coupon bond" as the coupon κ_t is the same in all the periods that the bond is alive. Coupons, however, may differ across issuance dates and they may depend on the shocks g^t. Section 5 is a special case when $\kappa_t = 0$ for all t.

[20] See Faraglia, Marcet, Oikonomou and Scott (2014b).

We normalize the payment at the end of the period to 1 unit of consumption. Therefore this 1-unit payment includes the principal $(1-\kappa_t)$ and the coupon paid in the period when the bond matures. This is a normalization, it simplifies formulas below.

To determine the size of the coupon we note that in US data long bonds trade at or close to par. In other words the debt management office designs coupons such that under current market conditions the bond price is very close to the principal, i.e. $q_t^N \approx 1-\kappa_t$.

The government budget constraint is now

$$q_t^N b_{N,t} = b_{N,t-N} + \sum_{j=1}^{N-1} b_{N,t-j}\kappa_{t-j} + g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)(A - x_t). \tag{55}$$

6.1 The Ramsey Program

The planner's objective is to maximize the agent's utility subject to (54), (55) and some ad hoc debt limits. The Lagrangian for the planner's program is now:

$$\begin{aligned} L = E_0 \sum \beta^t \Bigg\{ & u(c_t) + v(T - c_t - g_t) + \lambda_t \Bigg[b_{N,t}(\beta^N u_{c,t+N} + \sum_{j=1}^{N-1} u_{c,t+j}\kappa_t) \\ & -b_{N,t-N}u_{c,t} - \sum_{j=1}^{N-1} b_{N,t-j}\kappa_{t-j}u_{c,t} + S_t) \Bigg] + v_{1,t}(\widetilde{\overline{M}}_N - b_{N,t}) + v_{2,t}(b_{N,t} - \widetilde{\underline{M}}) \Bigg\} \end{aligned}$$

where the appropriate debt limits are :

$$b_{N,t} \in \left[\frac{\underline{M}}{\sum_{j=1}^{N-1}\beta^j + \kappa \sum_{j=1}^{N-1}\sum_{i=1}^{j}\beta^i}, \ \frac{\overline{M}}{\sum_{j=1}^{N-1}\beta^j + \kappa \sum_{j=1}^{N-1}\sum_{i=1}^{j}\beta^i} \right] \equiv [\widetilde{\underline{M}}, \ \widetilde{\overline{M}}] \tag{56}$$

for $\kappa = E(\kappa_t)$. As in the zero-coupon model the limits ensure that the steady state market value of debt is in $[\underline{M}, \ \overline{M}]$.

In the simulations we only consider cases when coupons are constant, i.e. $\kappa_t = \kappa$. From the above Lagrangian the first order condition for consumption is:

$$u_{c,t} - v_{x,t} + \lambda_t(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t+g_t) - v_{x,t}) + u_{cc,t}\kappa \sum_{j=1}^{N-1}(\lambda_{t-j} - \lambda_t)b_{N,t-j} + u_{cc,t}(\lambda_{t-N} - \lambda_t)b_{N,t-N} = 0 \tag{57}$$

and the condition for b_t^N:

$$\lambda_t E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j}\lambda_{t+j} + \beta^N u_{c,t+N}\lambda_{t+N}) + v_{2,t} - v_{1,t}. \tag{58}$$

The optimal policies again satisfy

$$\begin{aligned} \begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} &= F(g_t, \lambda_{t-1}, \ldots, \lambda_{t-N}, b_{N,t-1}, \ldots, b_{N,t-N}) \\ \lambda_{-1} &= \ldots = \lambda_{-N} = 0, \ \text{given } b_{N,-1}, ..., b_{N,-N}. \end{aligned}$$

The state vector includes the lags of the multiplier λ and all the lags of the bond quantities so that the dimensionality of the state vector is again $2N + 1$.

Even when the government issues non-zero coupon bonds the incentive to twist interest rates is present. This may seem surprising since the per period budget constraint in (55) is a function only of one price, the issuance price. However, bonds which haven't matured in t affect the governments intertemporal constraint and its future income and financing needs so that the government will be interested in twisting that price as well.

In the case of coupon bonds the government has the incentive to promise tax cuts in all periods from $t = 1$ to $N - 1$. Moreover, from (57) we can identify the term $\mathcal{D}_t$:

$$\mathcal{D}_t = \kappa \sum_{j=1}^{N} (\lambda_{t-j} - \lambda_t) b_{t-j}^N + (\lambda_{t-N} - \lambda_t) b_{t-N}^N$$

which drives interest rate twisting. This highlights that not only the level of debt issued from $t-1$ to $t-N$ matters (as in the case of zero coupon long bonds and no-buyback) but also the coupon payments matter to pin down the allocations in each period and in particular the level of taxation. For instance, in the case of a constant coupon bond and no buy back we have that (58) follows a complicated pattern which is a function of all the future terms $u_{c,t+j}\lambda_{t+j}$ for $j = 1, 2, \ldots, N$ weighted by the promised payments.

6.2 Some Analytic Results

We already found in Section 5 that long bonds under no-buyback may generate undesired tax volatility. We now show that this effect is alleviated if bonds pay sufficiently high coupons. However, in that case the bond positions are likely to be very volatile. We start with a general result.

Sufficiency of Measurability Conditions

We have already pointed out in Section 5.1 that under no-buyback and zero coupons, equilibrium constraints *cannot* be summarized in a standard implementability condition, hence the complete market allocation can not be achieved even under certainty. Now we explore in more generality the issue of how to write down a set of sufficient implementability conditions by considering coupon payments in a model with uncertainty.

Consider a feasible sequence of consumption $\{c_t\}$ and, associated with such a sequence, define the discounted sum of surpluses z_t as

$$z_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \left[\left(1 - \frac{v_{x,t+j}}{u_{c,t+j}} \right) (c_{t+j} + g_{t+j}) - g_{t+j} \right].$$

The literature has so far focused on finding sufficient implementability conditions for three separate types of government bonds - complete markets, incomplete markets and effectively complete markets. In these cases we have the following *standard results:* it is well known that under complete markets, where a full range of state contingent securities, exists a necessary and sufficient condition for equilibrium is that

$$z_0 = b_{-1} \tag{59}$$

(see, for example, Lucas and Stokey (1983) and Chari, Christiano and Kehoe (1991)). If by contrast markets are incomplete and consist of only a real riskless one-period bond (N=1) Aiyagari et al. (2002) show that, in addition to (59), the following measurability conditions are needed for a set of sufficient conditions

$$z_t \text{ is a function of } g^{t-1} \text{ for all } t > 0. \tag{60}$$

A $\{c_t\}$ that satisfies these conditions is supported by a sequence of bonds $z_t = b_{1,t-1}$ for all $t > 0$. Finally Angeletos (2002) extends this result to the case of multiple riskless bonds when long bonds are bought back one period after issuance (as in our Section 3) and assuming that there are enough bonds to effectively complete the markets. For simplicity we only state this result for the case where g takes two values and there are two bonds, a one- and an N-period bond denoted $b_{1,t}, b_{N,t}$. Angeletos shows the sufficient equilibrium conditions are :

$$z_t = \boldsymbol{b}_{1,t-1} + E_t\left(\beta^N \frac{u_{c,t+N}}{u_{c,t}}\right) \boldsymbol{b}_{N,t-1} \tag{61}$$

for random variables $\boldsymbol{b}_{i,t}$ measurable with respect to g^t for all $t \geq 0$ $i = 1, N$. In this case the equilibrium is supported by bond positions $\boldsymbol{b}_{1,t} = b_{1,t}$ and $\boldsymbol{b}_{N,t} = b_{N,t}$ for all $t > 0$.

All three of (59), (60), and (61) share the following feature: sufficient equilibrium conditions require that private wealth in all periods must equal the discounted sum of primary surpluses z_t. We now show that when bonds are as in the current section the analog condition

$$\sum_{j=1}^{N-1} b_{N,t-j}\left(\kappa_{t-j} + q_t^{N-j}\right) + b_{N,t-N} = z_t \text{ for all } t \tag{62}$$

is *not* sufficient for an equilibrium, where q_t^{N-j} is as in (54) with maturity $N - j$ and coupon κ_{t-j} (instead of κ_t).

To prove our point it is enough to show one case where (62) is not sufficient. We consider $N = 2$ so that (62) becomes

$$b_{2,t-1}\left(\kappa_{t-1} + \beta E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\right)\right) + b_{2,t-2} = z_t. \tag{63}$$

The example in Section 5.1 showed that for $\kappa = 0$ this equation is not sufficient. One may think that removing the constraint $\kappa = 0$ could make (63) sufficient. Indeed, if coupons are contingent on information available after the date of issuance markets, can be effectively completed. But contingent coupons are easily ruled out due to issues of moral hazard and because the fluctuations in coupons needed to complete the markets would be very large.[21] Therefore we only consider fixed coupons in the remainder of the section, where κ_t is determined at the date of bond issuance t and $b_{N,t}$ pays the same coupon during all periods $t + 1, \ldots, t + N$.

[21] In particular, κ_{t-1} can always be chosen contingent on g_t so as to guarantee that (63) always holds for a "properly designed" coupon. This can be done as follows. A constant bond issuance $b_{2,t} = b_{2,-1}$ is implemented if κ_{t-1} contingent on g_t can be chosen to satisfy

$$b_{2,-1} = \frac{z_t}{1 + \kappa_{t-1} + \beta E_t(\frac{u_{c,t+N}}{u_{c,t}})}$$

for all g^t so that (63) is certain to hold. In this case we are back to the standard case where (59) is the only implementability constraint. Obviously such coupons would have very large fluctuations as they would have to match fluctuations in the discounted sum z_t.

We can offer the following set of sufficient conditions

Result 4. *Assume $N = 2$, fixed coupons $\{\kappa_t\}$, no-buyback. Consider a consumption sequence $\{c_t\}$ such that the associated discounted sums of surpluses z_t satisfies*

1. $z_0 = b_{1,-1}(q_0^1 + \kappa_{-1}) + b_{2,-2}$ *for given initial conditions* $b_{1,-1}, b_{2,-2}$, κ_{-1};
2. *(63) for all $t > 0$ as for some random variables $b_{2,t}(g^t)$.*

If, in addition we have the following boundedness condition:

3. *the random variables $b_{2,t}(g^t)$ mentioned in 2. satisfy bounds (56) for sufficiently large $\underline{M}$, $\overline{M}$ for all t a.s.*

   then $\{c_t\}$ is a competitive equilibrium.

The *standard results* mentioned above focused on conditions 1. and 2.. The main reason to write the above result is to highlight that under no-buyback the boundedness condition 3. is also needed under no-buyback. It will turn out that condition 3 does not hold in many cases.[22] The proof follows a usual pattern and we do not offer details.[23]

A reference point will be coupons that are, roughly speaking, close to the net real rate of interest, so that bonds trade at (or close to) par. It follows from (54) that a coupon

$$\kappa_t^P = \frac{1 - \beta^2 E_t\left(\frac{u_{c,t+2}}{u_{c,t}}\right)}{1 + \beta E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\right)} \tag{64}$$

causes the bond price to trade at par, namely $q_t^2 = 1 - \kappa_t^P$.

Now, (63) can be rewritten as

$$\begin{aligned} b_{2,t-1} &= -z_t\delta_t + \delta_t b_{2,t-2} \\ \delta_t &= -\left(\kappa_{t-1} + \beta E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\right)\right)^{-1}. \end{aligned} \tag{65}$$

Since this is similar to a first order stochastic difference equation in $b_{2,t}$ it should be clear that if $|\delta_t| < 1$ then it is "more likely" that a $b_{2,t}$ satisfying (65) does not explode and, therefore, it satisfies the boundedness condition 3 of Result 4.

This shows in a generic way that high coupons help to smooth taxes, since a high κ_{t-1} drives $|\delta_t|$ below 1. But the boundedness condition 3. fails if coupons are small hence, in that case, the standard measurability conditions 1. and 2. of Result 4 are not sufficient.

We used the word "generic" in the last paragraph because there is always a configuration of initial conditions where the boundedness condition 3. holds even for low coupons: given a consumption

[22] In some cases fixed coupons can complete the markets and they can satisfy condition 3. For example, the portfolios of Angeletos, Buera and Nicolini can be implemented with a fixed κ_{t-1}, but this would require a very large and negative coupon. For the calibrated case of Buera and Nicolini a coupon of about minus 200% would implement the complete market allocation with a constant level of bonds. Again, we find this case of little interest as governments can not offer huge negative coupons.

[23] First prove necessity of (63). Then prove that if $b_{2,t}$ satisfies (63) the period-by-period budget constraints must hold.

sequence and a level of initial wealth $b_{1,-1}+(\beta+\kappa)b_{2,-2}$ there is always a value of $b_{1,-1}$ that guarantees that bond positions do not go to infinity even if $|\delta| > 1$, this would be an initial condition that satisfies a standard ending condition in difference equations, but for all other values of $b_{1,-1}$ and same wealth bonds go to infinity if $|\delta| > 1$.

The following two results make this generic idea concrete for some special cases.

Result 5. *Consider, as in Section 5.1 the case of $g_t = \overline{g}$ and constant taxes. The boundedness condition 3. is satisfied (generically) if and only if the bond trades at a price higher than par, that is*

$$\kappa_t = \kappa \geq \kappa^P = 1 - \beta. \tag{66}$$

It is clear that in this case $z_t = \overline{z}$ and $|\delta_t| = (\kappa + \beta)^{-1}$ so that if (66) holds then $b_{2,t}$ in (65) does not explode and boundedness condition 3. is satisfied. But for a low coupon $\kappa < \kappa^P = 1 - \beta$ then $b_{2,t-1}$ explodes and bonds violate any finite limit. The example of Section 5.1 is a special case of this result for $\kappa = 0$.

If bonds trade at par $\kappa_t^P = 1 - \beta$ (63) gives

$$b_{2,t-1} = \overline{z} - b_{2,t-2}$$

and b oscillates in a two-period cycle: $b_{2,t} = b_{2,t-2}$.

The next result shows a partial generalization to the case of uncertainty. It says that if coupons are sufficiently high long bonds and short bonds can implement the same allocations, but with low coupons long bonds implement fewer equilibria. Therefore one would expect less opportunities for tax smoothing in the presence of long bonds with low coupons. For this result we make the following assumptions:

A1 - $u(c) = c$;

A2 - *g_t iid, stochastic and a.s. bounded: Prob$(|g_t| < K^g) = 1$ for some $K^g < \mathcal{M}$ where* $\mathcal{M} \equiv \max_x(1 - v_x)(A - x)$.

Notice that with this utility $1 - v_x = \tau$, therefore $\mathcal{M}$ represents the maximum tax revenue that can be generated in a given period in equilibrium (the maximum of the Laffer curve).

Define $\mathcal{CE}_\kappa^N$ as the set of all competitive equilibrium allocations $\{c_t\}_{t=0}^{\infty}$ for long bonds of maturity N, with a constant coupon κ. With this notation Result 4. can be restated as saying that an allocation belong to $\mathcal{CE}_\kappa^N$ if and only if it satisfies conditions 1,2,3. For short bonds we write $\mathcal{CE}^1$ as the coupon is irrelevant.

Result 6. *Assume A1 and A2 above. Consider two identical economies, the first economy has a short bond $N = 1$ and the second $N = 2$ without buyback and coupon κ. Both economies have identical initial wealth, $b_{1,-1} = b_{2,-1}(\kappa + \beta) + b_{2,-2}$.*

a) If long bonds sell at higher than par, namely $\kappa > \kappa^P = 1 - \beta$, then $C\mathcal{E}^1 = C\mathcal{E}_\kappa^2$.

b) For zero coupons $C\mathcal{E}^1$ is strictly larger than $C\mathcal{E}_0^2$ (i.e. $C\mathcal{E}_0^2 \subset\neq C\mathcal{E}^1$).

Proof.

We first show that $\mathcal{CE}^2_\kappa \subset \mathcal{CE}^1$ for any coupon κ. Consider a given allocation $\{c_t\}_{t=0}^\infty \in \mathcal{CE}^2_\kappa$, with associated discounted sum of surpluses z_t and let $\{b_{2,t}\}$ the bond sequence that implements this equilibrium with 2−period bonds. Since $z_t = b_{2,t-1}(\kappa+\beta) + b_{2,t-2}$ it is clear that z_t is measurable with respect to information at $t-1$. It follows from proposition 1 in Aiyagari et al. (2002) that this allocation is also an equilibrium for $N = 1$ with $b_{1,t-1} = z_t$. Obviously, since $b_{2,t}$ is uniformly bounded so is $b_{1,t}$. Therefore $\mathcal{CE}^2_\kappa \subset \mathcal{CE}^1$.

All that remains for part *a)* is to show $\mathcal{CE}^1 \subset \mathcal{CE}^2_\kappa$ for sufficiently high κ. Given $\{c_t\}_{t=0}^\infty \in \mathcal{CE}^1$ and the corresponding bond allocation $b_{1,t}$ we construct the following $b_{2,t}$

$$b_{2,t} = -b_{2,t-1}\frac{1}{\beta+\kappa} + b_{1,t} = \sum_{j=0}^{t+1}\left(-\frac{1}{\beta+\kappa}\right)^j b_{1,t-j} + \left(-\frac{1}{\beta+\kappa}\right)^{t+2} b_{2,-2}.$$

Clearly, since $b_{1,t-j}$ is uniformly bounded and $\frac{1}{\beta+\kappa} < 1$ this $b_{2,t}$ satisfies (63) and the boundedness condition 3. This proves part a).

To show *b)* we construct one allocation in $\mathcal{CE}^1$ that is not in $\mathcal{CE}^2_0$. Consider the case $b_{1,-1} = 0$. Fix parameters $\alpha, \eta > 0$. Consider a policy such that given the state variables $(g_t, b_{1,t-1})$ tax revenue at t is given by

$$(1 - v_{x,t})(A - x_t) = E(g_t) + \alpha\,(g_t - E(g_t)) + \eta b_{1,t-1}. \tag{67}$$

This obviously defines hours, consumption, etc. as a function of $(g_t, b_{1,t-1})$.

For this policy $\alpha \leq 1$ governs how much the deficit increases when g is higher than average, thus it governs how much of an adverse shock is absorbed by deficit and debt. If we set $\alpha = 1$ this leads to a balanced budget and no tax smoothing. However if $0 < \alpha < 1$ there is some tax smoothing, an adverse g causes a deficit and higher debt. Parameter η governs the effect of past debt on current primary deficit. We assume α, η are chosen so that the right side of (67) is lower than $\mathcal{M}$ so there is always an x_t that solves (67), more on this later.

The budget constraint implies that the corresponding bond sequence is

$$b_{1,t}\beta = (1-\eta)b_{1,t-1} + (1-\alpha)\left[g_t - E(g_t)\right]$$

so that

$$b_{1,t} = \sum_{j=0}^{t}\beta^{-j-1}(1-\eta)^j(1-\alpha)\left[g_{t-j} - E(g_t)\right].$$

If $\eta > 1-\beta$ then $b_{1,t}$ is bounded above by $\frac{1-\alpha}{\beta+\eta-1}\left[K - E(g_t)\right]$ so that (67) belongs to $\mathcal{CE}^1$ for any $\alpha \in (0,1)$ and $\eta > 1-\beta$ such that revenue is feasible. Furthermore, from this equation it is clear that there are many values of α, η guaranteeing that the right side of (67) is lower than $\mathcal{M}$, as required above.

Now we check that this allocation does not belong to $\mathcal{CE}^2_0$. To implement the allocation (67) with

$N=2$ and $\kappa=0$ we would need a $b_{2,t}$ such that $b_{1,t}=b_{2,t}\beta+b_{2,t-1}$ hence

$$\begin{aligned}
b_{2,t} &= \sum_{j=0}^{t}(-\beta)^{-j-1}b_{1,t-j}=\sum_{j=0}^{t}(-\beta)^{-j-1}\sum_{i=0}^{t-j}\beta^{-i-1}(1-\eta)^{i}(1-\alpha)\left[g_{t-j-i}-E(g_t)\right]\\
&= (-\beta)^{-t}\beta^{-1}\left[1+(\eta-1)+(\eta-1)^2+\ldots+(\eta-1)^t\right](1-\alpha)\left[g_0-E(g_t)\right]\\
&\quad +(-\beta)^{-t}\beta^{-1}\left[1+(\eta-1)+(\eta-1)^2+\ldots+(\eta-1)^{t-1}\right](1-\alpha)\left[g_1-E(g_t)\right]+\ldots\\
&= -(-\beta)^{-t-1}(1-\alpha)\sum_{j=0}^{t}\frac{1-(\eta-1)^{j+1}}{2-\eta}\left[g_{t-j}-E(g_t)\right].
\end{aligned}$$

Now

$$\begin{aligned}
var(b_{2,t}) &= (-\beta)^{-2t-2}(1-\alpha)^2(2-\eta)^{-2}\left(\sum_{j=0}^{t}\left[1-(\eta-1)^{j+1}\right]\right)^2 var(g_t)\\
&> (-\beta)^{-2t-4}(1-\alpha)^2(2-\eta)^{-2}\eta^2\, var(g_t)
\end{aligned}$$

where the inequality comes from $\left|1-(\eta-1)^{j+1}\right|>\eta$ for all j.

Since $\beta<1$ then $(-\beta)^{-2t-4}\to\infty$ and $var(b_{2,t})\to\infty$ as $t\to\infty$, therefore any bond limits will be violated eventually and we can not find a $b_{2,t}$ that implements the policy (67). ■

Notice that, in order to obtain a sharp analytic result we had to assume linear utility. For a risk-averse u, the standard fiscal insurance effect of Angeletos, Buera and Nicolini would be present and long bonds would help to smooth taxes. In a standard calibration with risk aversion and no-buyback both effects will be present and the issue can only be resolved by numerical simulations, as we do below.

Volatility of positions

The previous results suggest that selling long bonds at par may be enough to smooth taxes, but this could be a bit too optimistic. The fact is that long bond positions that complete the markets can be very volatile. To see this, take the case used in Result 5. If bonds are sold at par the optimal solution implies

$$b_{2,t}=\overline{z}-b_{2,t-1}. \tag{68}$$

Therefore bonds display a two-period cycle $b_{2,t}=b_{2,t-2}$ for all $t\geq 2$. Similarly, in a model with uncertainty as in Result 6. and coupons at par, a shock in say, an even period, would cause higher debt in all future even periods and not in odd periods, this imparting volatility across even and odd periods.

Such fluctuations of bond positions would cause large variations in gross issuance of debt from one period to the next when initial conditions are very asymmetric, as in the war calibration of section 5.4.2. In the current setting this can be a problem because it makes it more likely that the bond limits are binding in many periods and, therefore, tax volatility arises. In general, high variability in gross bond issuance is often seen as undesirable in actual debt management practice.

Summary

A summary of all these results is that long bonds without buyback impart rollover cycles of periodicity N that cause taxes to be volatile. Coupons alleviate the problem but they may introduce large oscillations in gross bond issuance. If these oscillations are ruled out long bonds will not complete the markets even with coupons near par.

6.3 Simulations with coupons

6.3.1 Business cycle fluctuations

To see in detail the impact of paying coupons under uncertainty we need to resort once more to simulations. Consider again the case of Figure 4 based on persistent shocks, a ten year bond and positive levels of initial debt. In our simulations we set $\kappa_t = \kappa = 1 - \beta$ for all t, corresponding to coupons that trade approximately at par (exactly at par only in the risk neutral case).

Figure 4 shows the response of taxes to an adverse expenditure shock. In Section 5 we showed that no buy back induces greater volatility in taxes and produces a N period cycle. However as Figure 4 shows paying coupons produces less volatile N cycles. The intuition is that coupons spread the timing of cash payments from a bond and so reduce the magnitude of the N cycles.

Results 5. and 6. showed that in some cases coupons help sustain the same tax profile as with short bonds. Figure 4 does not show such an extreme case, we still find rollover cycles, unlike the case of short bonds $N = 1$, but the cycles are less pronounced than with zero coupons. What happens is that the bond limits we impose make it impossible for the government to smooth taxes as in short bonds because, as we explained in Section 6.2, this would cause a large variation in bond positions in order for coupons to achieve tax smoothing as with short bonds and bond limits will bind more often. Therefore the case with coupons is somewhere between zero-coupon long bonds and short bonds.

Coupons are essentially short term debt and taxes can now be raised in all periods from $t + 1$ to $t + N$ to finance the deficit caused by a high g_t. This suggests a fairly immediate explanation for why long term bonds pay coupons, governments can use coupons as a way of reducing tax volatility. This is confirmed by comparing Tables 3 and 6 where paying coupons under no buy back reduces the volatility of taxes and consumption compared to the case of no coupons and no buy back. It is even further confirmed by the war calibration.

[Table 6 About Here.]

6.3.2 Tax volatility after a war

Using the "end-of-war" calibration as in Section 5.4.2 in the model with coupons gives the moments reported in the last column of Table 5. We can see that there is still a very large tax volatility compared with buyback but that coupons at par do alleviate the rollover cycles. Again, modelling repurchase of long bonds and coupons explicitly is important, and the results give a reason for coupon payments.

6.4 Coupons, Commitment and Independent Powers

Having extended our model to allow for the empirically motivated features of no buy back and coupons we return again to considering the role of commitment in optimal debt management and fiscal policy. To that end, Figures 5 and 7 report the impulse responses of taxes and other model variables to a one standard deviation fiscal shock under both full commitment and our model of independent powers. In both cases we assume that $N = 10$, in Figure 5 we study the case of zero initial debt and in 7 positive initial debt. Figure 6 considers different maturities N. These figures show the oscillations in tax rates, with evident N period cycles, these arise for reasons explained in the previous section.

As hinted by inequalities (48) and the discussion in Section 5.3 we expect that if the government is in debt the role of commitment is to lower taxes for the next N periods after the shock. Figure 7 shows the responses when initial government debt is positive and $b_{-1} = .. = b_{-N}$. The government now wishes to reduce taxes between $t+1$ and $t+N$ to reduce the value of the initial debt burden in response to a high g_t. This causes a mild drop in tax response in these initial periods relative to the case of zero initial debt in Figure 5. The intuition for this has already been given around equation (48) and Section 5.3 so we do not repeat it here. In contrast, in an independent power model where the tax schedule is a function only of the exogenous shock and the level of debt we do not see the lower taxes in the first N periods when there is debt.

However the effect of interest rates twisting is overpowered by the considerably larger volatility originated by the N cycle property.

[Figures 5, 7 and 6 About Here.]

Figure 6 extends these results to different maturities.[24]

In Tables 6 and 7 we show sample moments generated by the simulations from the models of this section. The basic patterns of taxes, consumption and the market value of debt are similar to the ones generated by the model of the previous section. Debt is negative in the longer run because governments wish to accumulate precautionary savings for tax smoothing purposes. The additional volatility of taxes generated by independent powers however is larger now.

[Table 7 About Here.]

To conclude, long bonds under no-buyback generate tax volatility causing spikes in the tax rates at redemption dates. These patterns are mitigated under commitment as the tax smoothing objective, summarized in the multipliers, influences the policy functions. Coupons help reduce the lumpiness and volatility of tax rates. The volatility of taxes can be very high if bonds issued in the last N periods have very different sizes, as it would happen after a war or a very deep recession.

7 Conclusions

In contrast to the case of short bonds, analysing long bonds requires numerous assumptions about the institutional setup. We have considered two such assumptions here - does the government buyback each period all outstanding government debt? do government bonds pay coupons?- These issues have not been addressed directly in the literature on long bonds, but we find that they matter considerably for the behaviour of taxes and debt.

[24] Clearly the N cycle property of the tax schedule coincides with the maturity date of debt in the figure. Across all values for N considered the variability of taxes in independent powers prevails.

Our focus has been to understand how a government issuing long bonds impacts on optimal fiscal policy. It is well known in the literature that long bonds offer fiscal insurance in the sense that in response to adverse expenditure shocks their price declines. However we have shown that over and above fiscal insurance there are additional intertemporal channels - interest rate twisting and rollover cycles - that create additional tax volatility. The closer the assumptions we make regarding the structure of long bonds are to those we observe in practice (especially around no repurchase before maturity) the more important these additional channels are. Thus whilst long bonds may provide fiscal insurance they also may induce additional tax volatility lessening their attractiveness and suggesting a role for short term debt - an issue we examine in Faraglia, Marcet, Oikonomou and Scott (2014b)- and for callable bonds. Further not only is the attractiveness of long bonds offset by this additional tax volatility but we find cases where long bonds are incapable of completing the market even under certainty.

We have also considered the nature of commitment that the government faces under optimal debt management. In the standard Ramsey institutional set up we have shown that the role of commitment for optimal policy under incomplete markets is a repeated attempt at lowering current interest rates by promising future tax cuts, depending on the current shock. It is this feature that makes modelling long bonds so computationally demanding and which motivates us to suggest an alternative institutional set up (independent powers) that is relatively easy to solve and that displays clearly the relevance of commitment. This modelling approach has applicability in a wider class of models than public finance.

We show our insights both through analytical examples and simulations. Our analytic examples provide insight and intuition to the mechanisms at work with long bonds and these are borne out in our simulations. With no-buyback, taxes and outcomes are more volatile than with buyback or short bonds, coupon payments help reduce this volatility. However many of our analytic examples have a form that emphasizes different factors from our business cycle focused simulations. In the face of one off large shocks (such as wars or financial crises) some of the complexities of long bonds that we highlight become very important. For instance, the impossibility result (that long bonds may not complete the market) requires an uneven debt structure before the current period, an outcome that is unlikely to occur in our simulations of business cycles with highly persistent shocks but that happens naturally after a few very large deficits, such as caused by wars or deep recessions. This suggests that having some flexibility with buyback is important after these events. This provides some grounds for the use of short bonds and/or callable bonds, an issue that we leave for future research.

The existing literature has taken a normative approach to debt management. It has assumed a certain structure for bonds and shown how under this structure long bonds excel in providing fiscal insurance and are key to optimal debt management. We examine different structures for long bonds and show that the ability to achieve fiscal insurance through long bonds is both reduced and offset by additional tax volatility making long bonds less attractive. An obvious response is to argue that governments should not engage in no-buyback. If governments simply repurchased all existing bonds regardless of maturity every period then the fiscal insurance benefits of long bonds would stand unalloyed. Whilst debt managers give many reasons why they do not repurchase every period and a few moments introspection can lead to many plausible theoretical candidates as to why they might not, it is indeed an important lacunae in our understanding of debt management as to why governments repurchase only occasionally.

However until we have a better understanding of the reasons why debt managers don't repurchase and whether or not it is optimal to do so we are wary of restricting assumptions about long bonds to the case of zero coupon and repurchase. Inevitably incomplete market models of debt management have to assume some market imperfections exogenously. Most of the existing literature for instance has assumed that bonds are simply risk free or that the government can issue bonds of only one period and that period coincides with the frequency of government shocks. Similarly assuming that the government buys back each and every bond every period is not grounded in any optimality and neither is our opposite extreme assumption that governments never buyback until maturity. Our assumption does however have the merit of being the closest to what we observe in practice and the fact that we show it matters for the optimality of long bonds in debt management suggests the analysis of maturity in government bonds is more complex than just the standard fiscal insurance channel.

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Table 1: **Moments: Commitment Model, Buyback**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.49	0.253	-0.233	-8.90	52.54	0.248	0.156	-15.98	52.58	0.245	0.317	-19.23
	10	52.50	0.252	-0.198	-7.18	52.54	0.248	0.120	-13.46	52.56	0.246	0.257	-16.28
	15	52.50	0.252	-0.184	-6.50	52.53	0.248	0.095	-11.76	52.55	0.247	0.209	-14.01
	20	52.50	0.252	-0.168	-5.83	52.52	0.249	0.076	-10.47	52.54	0.247	0.178	-12.48
std													
	5	2.27	0.0223	1.28	11.82	3.13	0.0357	1.58	23.82	3.48	0.0435	1.57	30.96
	10	2.27	0.0224	1.23	11.22	3.13	0.0357	1.59	24.23	3.48	0.0438	1.59	31.97
	15	2.27	0.0225	1.29	11.55	3.13	0.0357	1.60	24.86	3.48	0.0439	1.65	32.54
	20	2.27	0.0226	1.31	11.50	3.13	0.0357	1.61	25.03	3.48	0.0439	1.367	32.83

Notes: The table shows key moments from the optimal policy model with commitment under buyback. The first four rows plot the averages for consumption, taxes, deficit and the market value of debt in the case of four different maturities ($N = \{5, 10, 15, 20\}$). The last four rows show the standard deviations of these quantities. The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions for government debt are uniformly distributed over $[-\overline{M}, \overline{M}]$.

Table 2: **Moments: Independent Powers Model, Buyback**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.49	0.253	-0.226	-8.23	52.54	0.248	0.156	-15.88	52.58	0.245	0.323	-19.49
	10	52.50	0.252	-0.198	-6.95	52.53	0.248	0.120	-13.33	52.56	0.246	0.260	-16.40
	15	52.50	0.252	-0.178	-6.09	52.53	0.249	0.095	-11.57	52.55	0.247	0.215	-14.23
	20	52.50	0.252	-0.161	-5.34	52.52	0.249	0.076	-10.01	52.54	0.247	0.175	-12.31
std													
	5	2.29	0.0224	1.14	11.05	3.14	0.0357	1.47	23.45	3.48	0.0436	1.51	31.12
	10	2.29	0.0224	1.13	11.00	3.14	0.0357	1.48	23.93	3.49	0.0438	1.54	32.20
	15	2.29	0.0223	1.12	10.97	3.14	0.0357	1.48	24.17	3.49	0.0440	1.55	32.78
	20	2.29	0.0223	1.12	10.94	3.14	0.0356	1.48	24.34	3.49	0.0440	1.56	33.20

Notes: The table shows key moments from the independent power model under buyback. The moments reported were constructed using the same samples as in commitment model (Table 1).

Table 3: **Moments: Commitment Model, No-Buyback**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.48	0.253	-0.259	-9.93	52.57	0.247	0.184	-17.95	52.58	0.245	0.332	-20.22
	10	52.48	0.253	-0.230	-8.33	52.56	0.247	0.143	-15.22	52.56	0.246	0.267	-17.07
	15	52.49	0.252	-0.202	-7.16	52.55	0.247	0.121	-13.48	52.55	0.247	0.233	-15.24
	20	52.49	0.252	-0.194	-6.32	52.55	0.248	0.107	-12.28	52.55	0.247	0.213	-13.99
std													
	5	2.29	0.0220	1.29	12.37	3.13	0.0360	1.60	23.66	3.51	0.0445	1.63	30.99
	10	2.29	0.0222	1.30	12.62	3.13	0.0361	1.66	25.23	3.52	0.0451	1.72	34.54
	15	2.30	0.0233	1.24	12.38	3.14	0.0369	1.65	25.49	3.52	0.0459	1.73	34.54
	20	2.33	0.0246	1.14	11.93	3.16	0.0379	1.61	25.16	3.53	0.0468	1.70	34.51

Notes: The table shows key moments from the commitment model under no buyback (zero coupons). The table reports the moments over three different horizons: 1) 40 observations, 18000 samples. 2) 2000 observations, 18000 samples, and 3) 4500 observations 18000 samples (5000 observations were generated for each sample, 500 observations were dropped). The initial conditions are $b_{N,-1} = b_{N,-2} = ... = b_{N,-N}$ where $b_{N,-j}, j = 1,..,N$ are uniformly distributed in $\frac{1}{\sum_{i=1}^{N}}\beta^i[-\overline{M},\overline{M}]$.

Table 4: **Moments: Independent Powers Model, No-Buyback**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.50	0.252	-0.171	-5.36	52.55	0.248	0.089	-11.58	52.55	0.246	0.205	-13.88
	10	52.50	0.251	-0.151	-4.41	52.54	0.248	0.058	-9.39	52.53	0.248	0.148	-11.06
	15	52.51	0.251	-0.133	-3.72	52.54	0.249	0.0423	-8.03	52.53	0.248	0.119	-9.44
	20	52.51	0.251	-0.123	-3.19	52.53	0.249	0.031	-7.00	52.52	0.248	0.100	-8.27
std													
	5	2.32	0.0228	1.11	10.87	3.15	0.0356	1.47	24.37	3.52	0.0449	1.56	35.26
	10	2.31	0.0228	1.14	11.20	3.14	0.0359	1.53	25.25	3.52	0.0455	1.64	36.51
	15	2.33	0.0238	1.11	11.08	3.15	0.0368	1.53	25.11	3.54	0.0463	1.65	36.39
	20	2.34	0.0250	1.03	10.76	3.17	0.0379	1.51	24.61	3.55	0.0472	1.64	35.79

Notes: The table shows key moments from the independent power model under no buyback (zero coupons). The moments reported are constructed using the same samples as in Table 3.

Table 5: **Wars and Taxes**

Horizon		**Buyback**	**No Buyback**	
			Coupon=0	**Coupon at par**
20	mean	0.283	0.305	0.300
	std	0.025	0.101	0.077
40	mean	0.278	0.281	0.279
	std	0.028	0.078	0.062
60	mean	0.273	0.274	0.272
	std	0.031	0.068	0.056

Notes: Tax mean and standard deviation at different horizons after the war for different types of long bonds.

Table 6: **Moments: Commitment Model, No-Buyback & Coupons**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.47	0.253	-0.252	-9.63	52.57	0.247	0.175	-17.39	52.61	0.244	0.336	-20.36
	10	52.48	0.252	-0.225	-8.217	52.56	0.247	0.139	-14.98	52.59	0.245	0.278	-17.60
	15	52.49	0.252	-0.200	-7.10	52.55	0.247	0.114	-13.15	52.59	0.245	0.239	-15.57
	20	52.49	0.252	-0.186	-6.31	52.55	0.248	0.096	-11.78	52.58	0.246	0.211	-14.03
std													
	5	2.28	0.0218	1.29	12.29	3.13	0.0359	1.60	23.84	3.49	0.0441	1.63	30.93
	10	2.28	0.0216	1.30	12.51	3.13	0.0358	1.64	25.12	3.49	0.0445	1.70	33.22
	15	2.29	0.0221	1.25	12.30	3.13	0.0361	1.62	25.19	3.50	0.0448	1.68	33.58
	20	2.30	0.0228	1.19	11.99	3.14	0.0366	1.58	24.83	3.50	0.0452	1.64	33.29

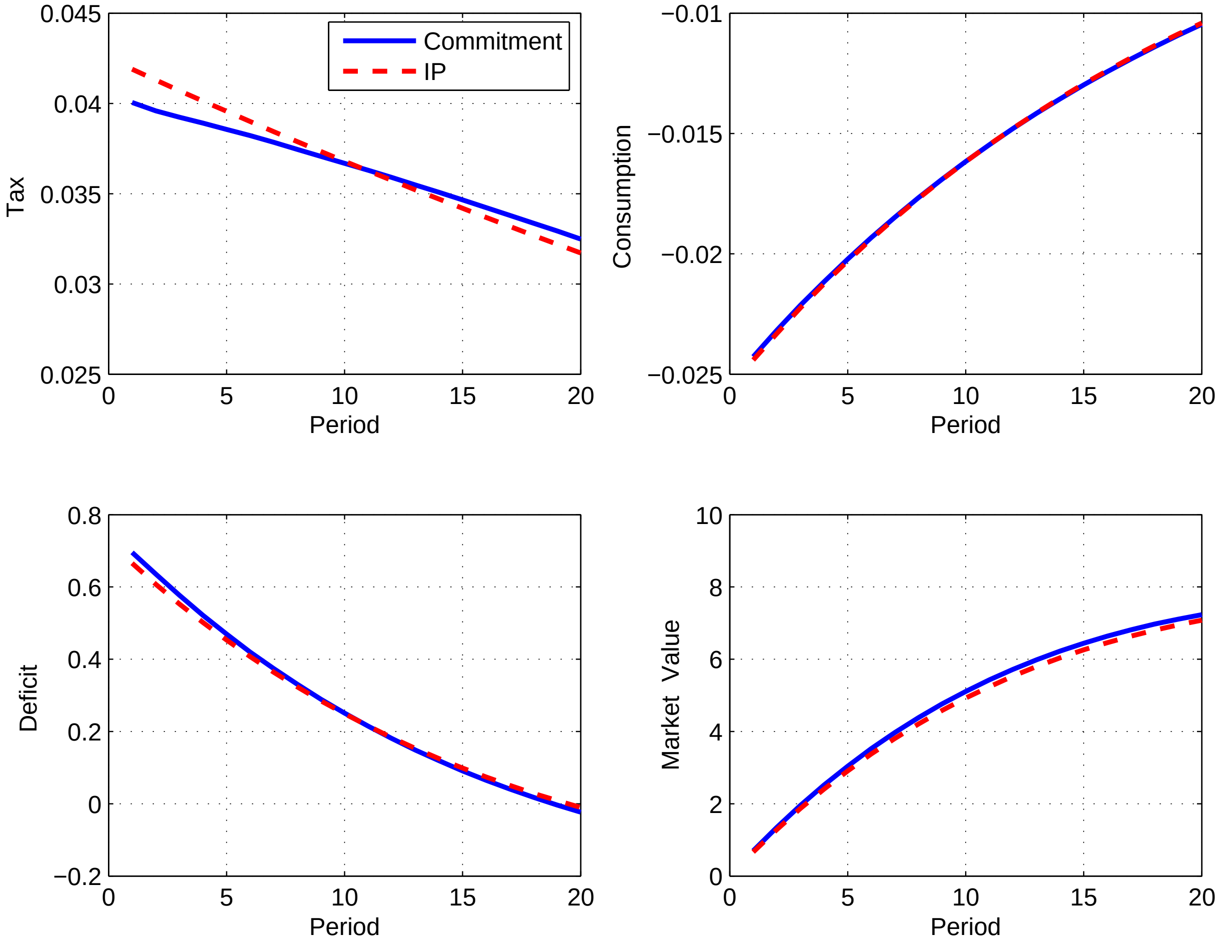
Notes: The table shows key moments from the commitment model with coupons and no buyback. The initial conditions for $b_{N,-j}, j = 1, 2, ..., N$ are uniformly distributed over $[-\widetilde{\overline{M}}, \widetilde{\overline{M}}]$. We further assume $b_{N,-1} = b_{N,-2} = ... = b_{N,-N}$. The choice of the number of samples and sample lengths is the same as in Table 1.

Table 7: **Moments: Independent Powers Model, No-Buyback & Coupons**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.49	0.252	-0.176	-5.67	52.55	0.248	0.092	-11.72	52.550	0.246	0.202	-13.76
	10	52.50	0.251	-0.150	-4.45	52.54	0.248	0.053	-9.07	52.53	0.248	0.134	-10.45
	15	52.50	0.251	-0.132	-3.70	52.53	0.249	0.035	-7.61	52.52	0.248	0.101	-8.73
	20	52.51	0.251	-0.122	-3.23	52.53	0.249	0.0234	-6.65	52.52	0.248	0.080	-7.59
std													
	5	2.31	0.0226	1.13	11.08	3.14	0.0356	1.49	24.24	3.52	0.0449	1.57	34.42
	10	2.30	0.0223	1.15	11.28	3.14	0.0355	1.53	25.01	3.52	0.0450	1.62	35.59
	15	2.31	0.0228	1.12	11.15	3.14	0.0359	1.51	24.98	3.52	0.0454	1.61	35.71
	20	2.33	0.0235	1.08	10.97	3.15	0.0364	1.49	24.71	3.54	0.0458	1.59	35.38

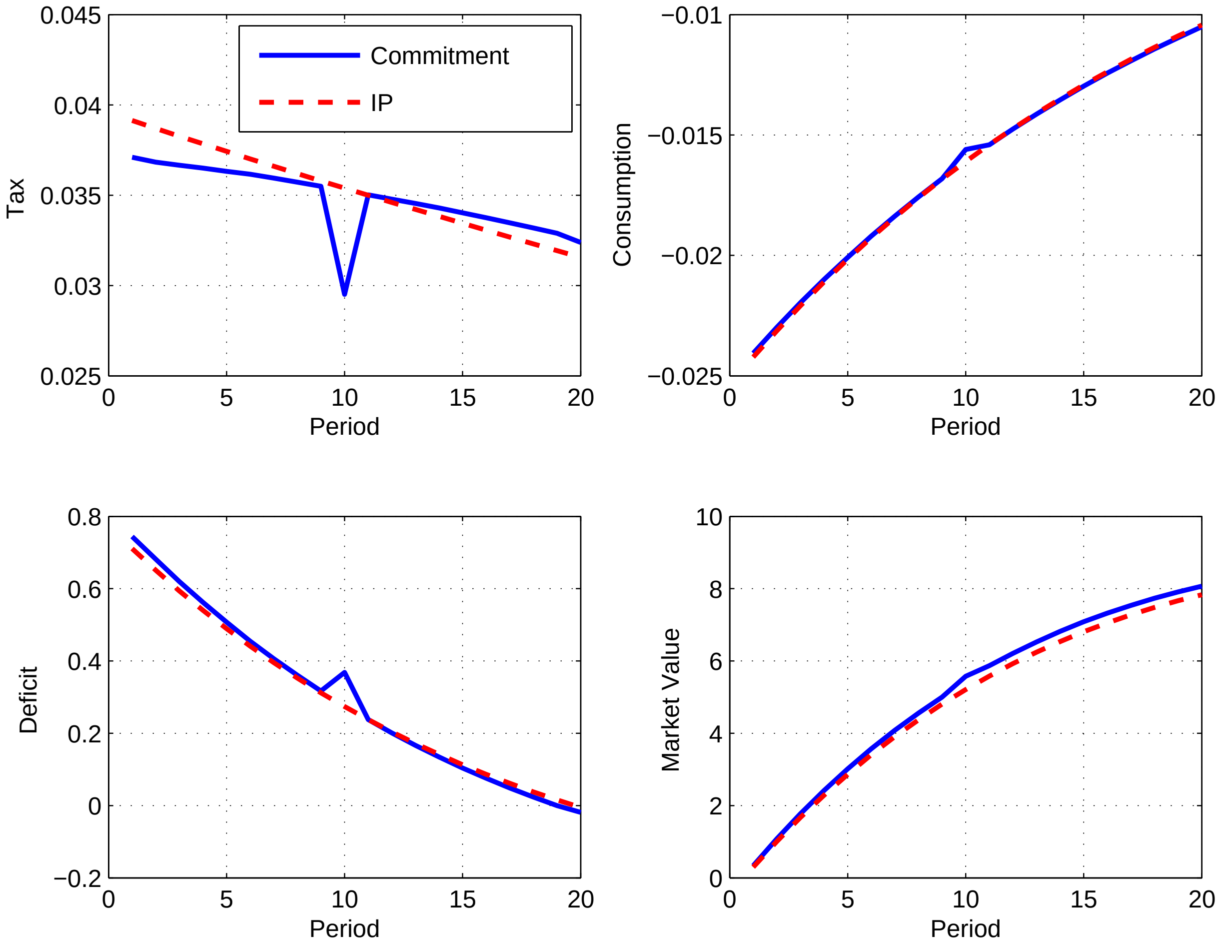
Notes: The table shows key moments from the independent power model with coupons and no buyback. The moments reported are constructed using the same samples as in Table 6.

Figure 1: **Impulse Responses under Zero Initial Debt: Buyback Model**



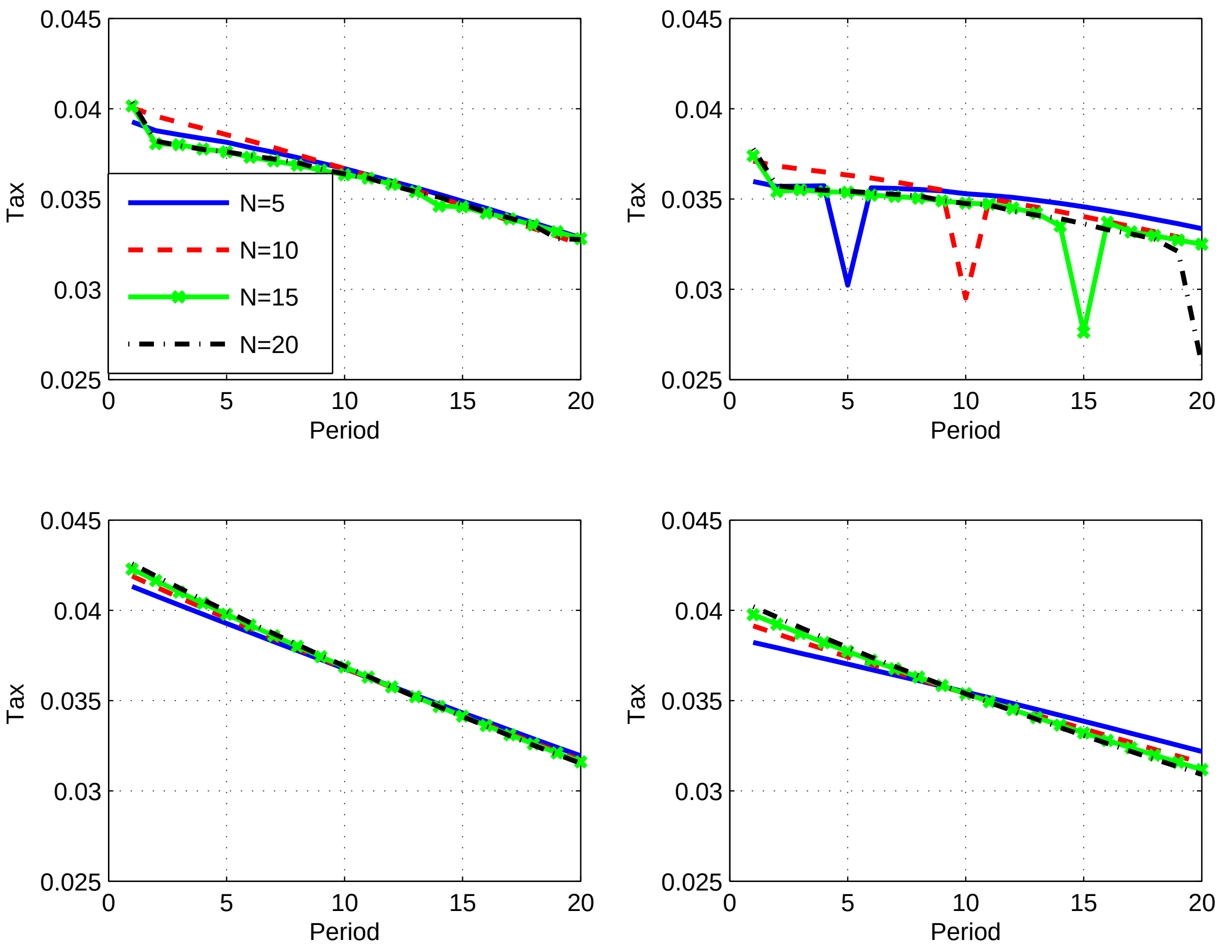
Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

Figure 2: **Impulse Responses under Positive Initial Debt: Buyback Model**



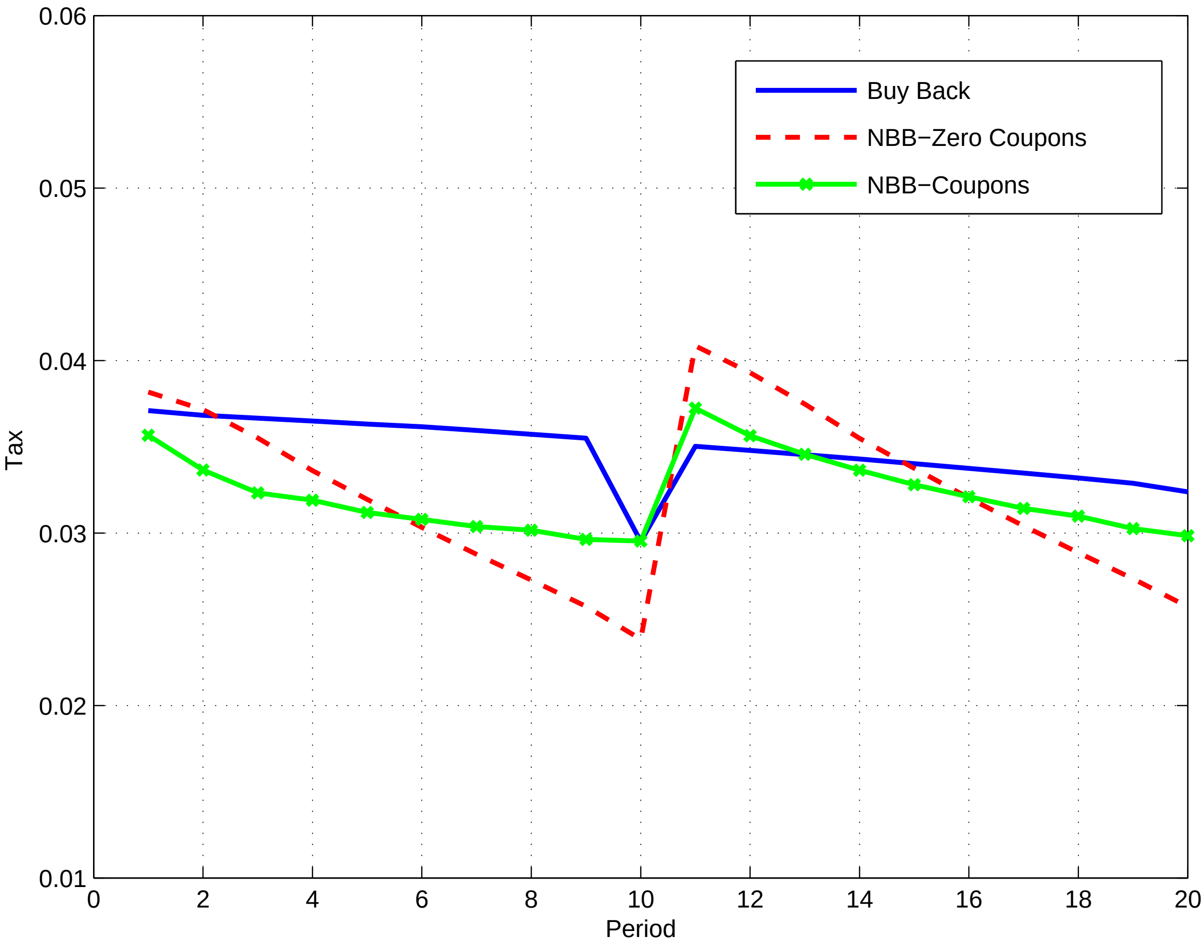
Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The initial debt level is 50
par

Figure 3: **Impulse Responses of Taxes: Buyback Model**



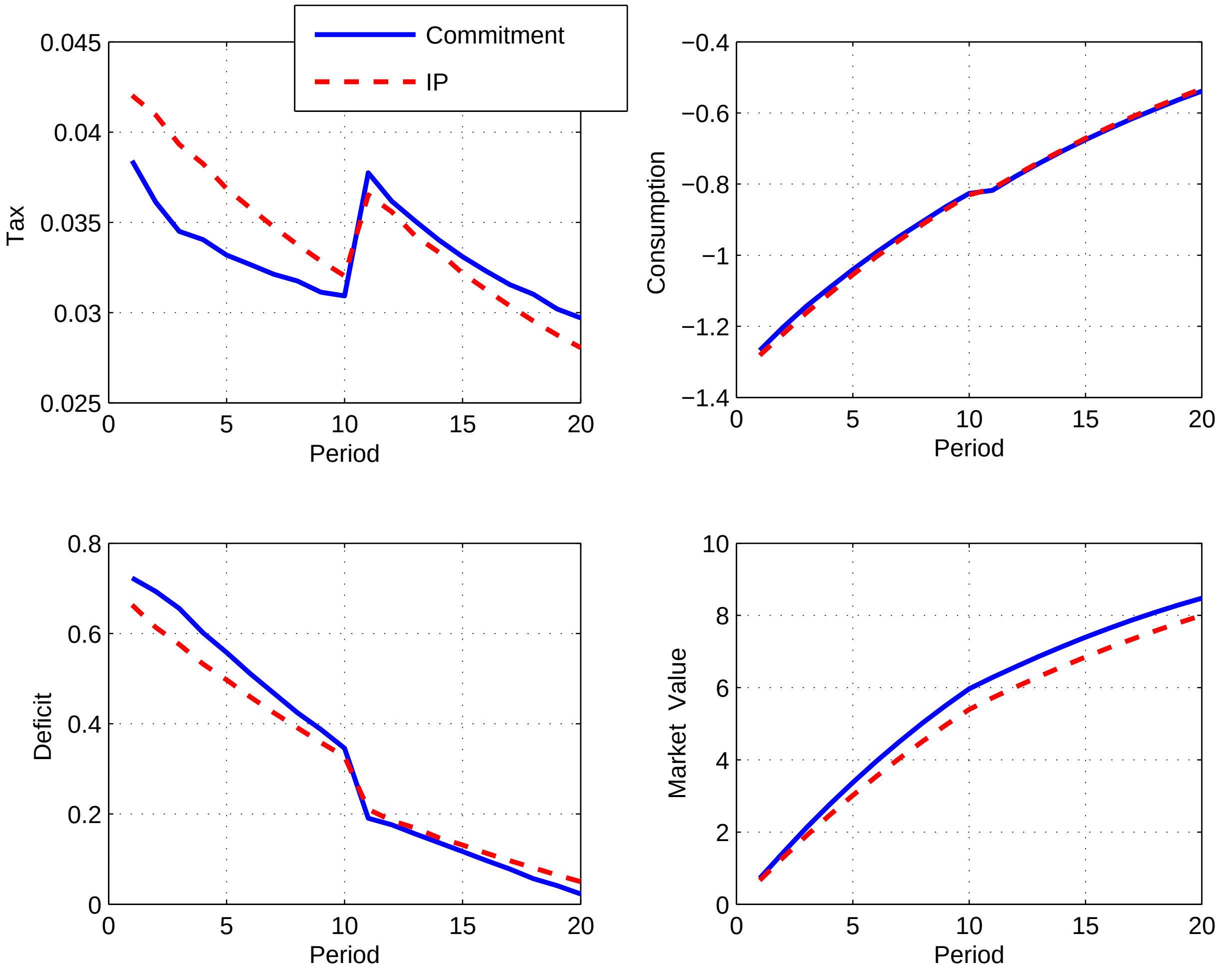
Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under buyback with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

Figure 4: **Impulse Response of Taxes: Commitment Models**



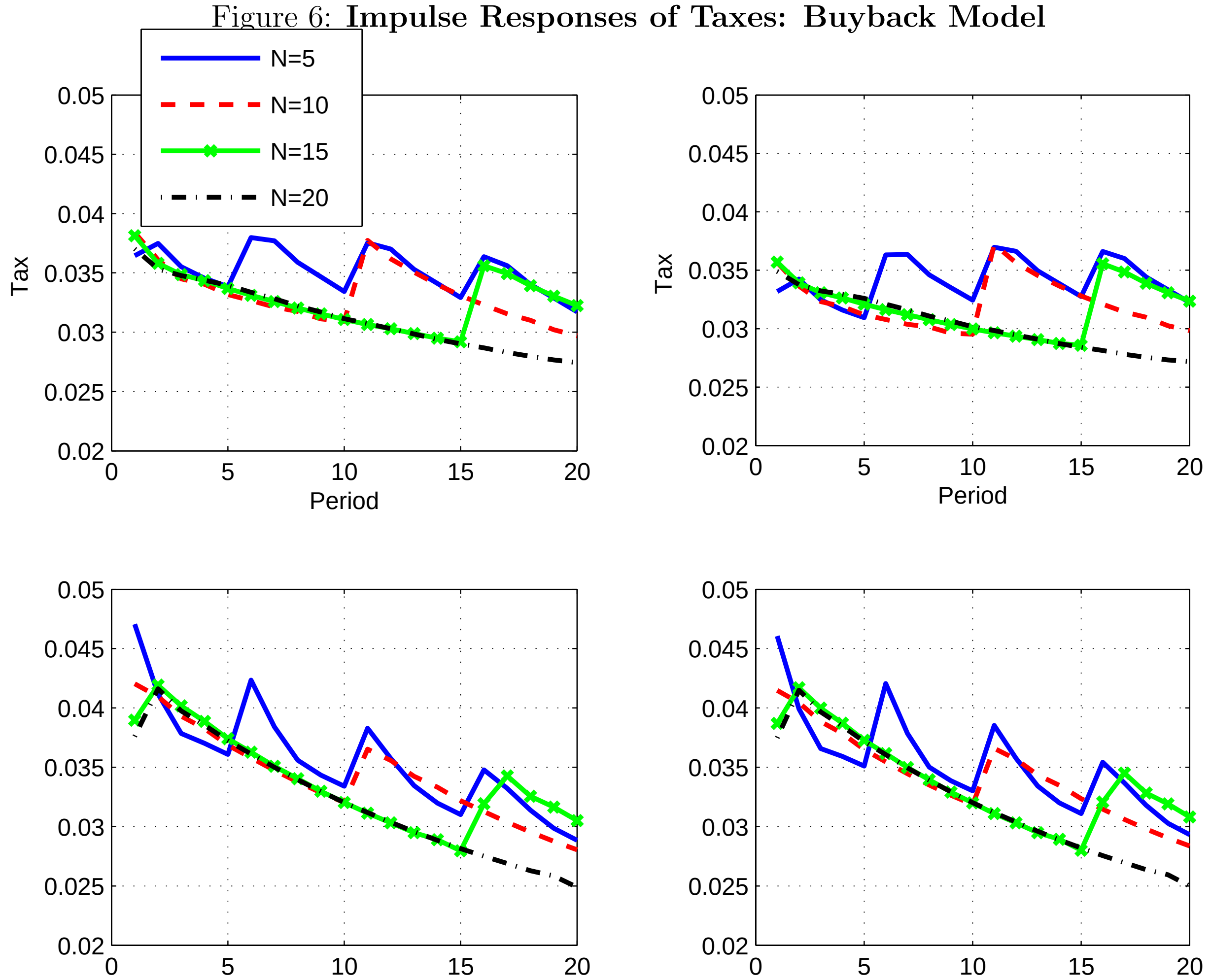
Notes: The Figure plots the impulse response of taxes to a spending shock. The solid line is the response under the assumption that debt is bought back in every period, the dashed line shows the case of no buyback and zero coupons. Finally, the crossed line shows the case of non zero coupon bonds.

Figure 5: **Responses under Zero Initial Debt: No-Buyback Model with Coupons**



Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

Figure 6: **Impulse Responses of Taxes: Buyback Model**



Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under decaying coupons with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

A Derivations- No Commitment Models

A.1 Independent Powers - Constant Coupons and no-buyback

Consider the first model of Section 3. In the no-commitment case (independent powers) we have the following Lagrangian:

$$\mathcal{L} = E_0 \sum \beta^t \left\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \left[b_t^N q_t^N u_{c,t} \ - b_{t-N}^N u_{c,t} - \kappa \sum_{j=1}^{N} b_{t-j}^N u_{c,t} + S_t) \right] \right.$$
$$\left. + v_{1,t}(\widetilde{\widetilde{M}}_N - b_t^N) + v_{2,t}^N (b_t^N - \underline{\widetilde{M}}_N) \right\}$$

and taking first order conditions we have:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} - u_{cc,t} \lambda_t [b_t^N q_t^N - \kappa \sum_{j=1}^{N} b_{t-j}^N - b_{t-N}^N] = 0$$

which through substitution of the budget constraint gives:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} - u_{cc,t} \lambda_t [g_t - (1 - \frac{v_{x,t}}{u_{c,t}})(c_t + g_t)] = 0.$$

Moreover, off corners the analogous condition b_t^N is:

$$\lambda_t E_t (\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t (\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}).$$

We utilize the above first order conditions to solve the optimal no-commitment model. Notice that in order to solve this model we have to apply the "Condensed PEA" methodology of FMOS (2014 b). The state vector for this model includes g_t $b_{N,t-j}$, $j = 1, 2, \ldots, N$. This gives $N + 1$ state variables.

B Appendix: Decaying Coupon Perpetuities

In order to overcome the problem of dimensionality some authors model long bonds as perpetuities with decaying coupon payments where the rates of decay mimic differences in maturity (e.g Woodford (2001), Broner, Lorenzoni and Schmulker (2013), Arellano and Ramanarayanan (2008), Chen, Curdia and Ferrero (2012)). Our independent powers model can be considered as an alternative means of achieving the same computational parsimony.

In this case governments issues perpetuities, b, with coupon payments that decay geometrically i.e. a bond with decay factor δ_L pays a coupon equal to $\delta_L^j b_j$ in period j. The decay rate determines effective bond maturity as duration is defined by $1/(1-\delta_L)$ so that a bond of effective maturity 10 years has $\delta_L = 0.1$. In this case total payments from all previously and currently issued perpetuities are then given by $B_t = b_t + \delta_L b_{t-1} + \delta_L^2 b_{t-2} + \ldots + \delta_L^t b_0$ which follows the recursive structure $B_t = \delta_L B_{t-1} + b_t$. Treating this as the outstanding stock of the perpetuity we have a convenient way of dealing with long maturity bonds which dramatically reduces the state space as it is only necessary to keep track of the total number of bonds issued and not the number of bonds issued in each period. This reduction in the state space means that the "Condensed PEA" is no longer required and the model can be solved using more conventional methods.

Whilst assuming decaying coupon payments has great computational merit it is not without modelling consequences. One justification for assuming decaying payoffs is that it mimics a bond portfolio with fixed shares that decay with maturity. This however does not seem to comply with the empirical evidence of US debt management whereby shares of long and short bonds are indeed time varying, though highly persistent and dont decline with maturity (see Section 2 in FMOS (2014 b)). Further, modelling bond payoffs in this way is contrary to the structure of most government portfolios where the majority of the payoff occurs at the time of maturity, as we have modelled in this paper, whereas with decaying coupons the majority of cash flow is paid out in the early years. Moreover if our goal is to build a model of debt management where the object is precisely to study the appropriate portfolio weights, assuming fixed portfolio weights would seem inappropriate. However this approach has been used in the literature and does offer substantial computational efficiency so it is worth comparing it with our own approaches.

Under perpetual bonds the government budget constraint becomes:

$$B_{t-1} = S_t/u_{c,t} + p_t\left(B_t - \delta_L B_{t-1}\right)$$

where $B_t - \delta_L B_{t-1} = b_t$ is the amount of bonds that the government issues in period t and B_{t-1} is the amount of coupons and maturing bonds that the government has to repay in the same period.

The Ramsey problem then becomes:

$$\max_{\{c_t, B_t, p_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\}$$

$$B_{t-1} = S_t/u_{c,t} + p_t (B_t - \delta_L B_{t-1})$$

$$p_t = \frac{\beta E_t (u_{c,t+1}(1 + \delta_L p_{t+1}))}{u_{c,t}}$$

$$\frac{\delta}{1-\delta\beta} b_{N,t} \in [\underline{M}, \overline{M}].$$

The price of the bond can be rewritten as $p_t = \frac{\beta E_t \left(\sum_{j=0}^{\infty} (\beta\delta_L)^{j-1} u_{c,t+j}\right)}{u_{c,t}}$, that shows that it is a function of all the future marginal utilities since the bond will pay an income for the rest of the time.

B.1 The Ramsey Program

We can rewrite the Lagrangian of the problem as:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t) + \lambda_t [S_t + u_{c,t} p_t (B_t - \delta_L B_{t-1}) - u_{c,t} B_{t-1}]$$
$$+ u_{c,t} (\mu_t p_t - \beta \mu_{t-1} (1 + \delta_L p_t))\}$$

dropping the debt limits for brevity.

The first order conditions of the decaying coupon model are as follows:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t})$$
$$+ u_{cc,t} [\lambda_t p_t (B_t - \delta_L B_{t-1}) - \lambda_t B_{t-1} + \mu_t p_t - \mu_{t-1} (1 + \delta_L p_t)] = 0 \tag{69}$$

$$\lambda_t u_{c,t} p_t = \beta E_t (\lambda_{t+1} u_{c,t+1} (1 + \delta_L p_{t+1})) \tag{70}$$

$$\mu_t = \delta_L \mu_{t-1} - \lambda_t (B_t - \delta_L B_{t-1}). \tag{71}$$

A new state variable, μ_t, emerges and the state space becomes $\{g_t, \mu_{t-1}, B_{t-1}\}$. Using (71) μ_t may be expressed as:

$$\mu_t = -\sum_{j=0}^{\infty} \delta_L^j \lambda_{t-j} (B_{t-j} - \delta_L B_{t-1-j}) \tag{72}$$

where μ_t is a function of all the past government promises, λ's. Equation (72) can be substituted in (69)

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t})$$
$$- u_{cc,t} \left[B_{t-1} \lambda_t - \sum_{j=0}^{\infty} \delta_L^j \lambda_{t-j-1} (B_{t-j-1} - \delta_L B_{t-2-j}) \right] = 0$$

which can be further rearranged into:

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + g_t\right) - v_{x,t}\right) - u_{cc,t} \left[\sum_{j=0}^{\infty} \delta_L^j (\lambda_{t-j} - \lambda_{t-j-1}) B_{t-j-1}\right] = 0 \quad (73)$$

According to (73) we have: $\mathcal{D}_t = \sum_{j=0}^{\infty} \delta_L^j (\lambda_{t-j} - \lambda_{t-j-1}) B_{t-j-1}$. This implies that interest rate twisting now concerns the entire history $j = 0, 1, 2, \ldots$ of issuances weighted by δ_L^j, since bonds payout coupons forever.

This result can be further simplified if we follow the same steps taken in Section 2. We can write the implementability constraint as:

$$E_t \sum_{j=0}^{\infty} \beta^j S_{t+j} = B_{t-1} E_t \sum_{j=0}^{\infty} \left(\delta_L \beta\right)^j u_{c,t+j} = \left(\sum_{i=1}^{t} \delta_L^{i-1} b_{t-i}\right) E_t \sum_{j=0}^{\infty} \left(\delta_L \beta\right)^j u_{c,t+j}.$$

If we assume no uncertainty this becomes:

$$\sum_{j=0}^{\infty} \beta^j S_{t+j} = \left(\sum_{i=1}^{t} \delta_L^{i-1} b_{t-i}\right) \sum_{j=0}^{\infty} \left(\delta_L \beta\right)^j u_{c,t+j}.$$

It becomes clear that the government has an incentive to affect the interest rates and consequently the taxes on an infinite horizon with decaying weights.[25] On the other hand under independent powers these terms have no direct influence on fiscal policy.

B.2 Independent Powers - Decaying Coupons and no-buyback

Consider now the second model of Section 3, the decaying coupon model. Under no-commitment we have:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{u\left(c_t\right) + v\left(x_t\right) + \lambda_t \left[S_t + u_{c,t} p_t \left(B_t - \delta_L B_{t-1}\right) - u_{c,t} B_{t-1}\right]\right\}$$

therefore, the planner is assumed to not control bond prices p_t and we have dropped for brevity the debt limits.

The first order conditions are as follows:

$$u_{c,t} - v_{x,t} + \lambda_t \left(u_{cc,t} c_t + u_{c,t} + v_{xx,t} \left(c_t + g_t\right) - v_{x,t}\right) + u_{cc,t} \left[\lambda_t p_t \left(B_t - \delta_L B_{t-1}\right) - \lambda_t B_{t-1}\right] = 0$$

and

$$\lambda_t u_{c,t} p_t = \beta E_t \left(\lambda_{t+1} u_{c,t+1} \left(1 + \delta_L p_{t+1}\right)\right).$$

The appropriate state variables for this model are obviously B_{t-1}, g_t.

[25] This is exactly the same as the case of a model with no buyback. The budget constraint there becomes:

$$\sum_{j=0}^{\infty} \beta^j S_{t+j} = \sum_{i=0}^{N} p_{N-i,t} b_{N,t-i}.$$

B.3 Simulation Results

We now consider the properties of the numerical solution to the decaying coupon model. In Figures 7 , 8 and 9 we plot the responses of taxes and other key model variables in the case of decaying coupons under both commitment and independent powers.

[Figures 7 , 8 and 9 About Here]

Following a spending shock tax rates are now much smoother than the ones generated by the constant coupons and no-buyback model analysed in the text, even though the assumption of no-buyback is maintained. This smoothness (relative to the responses shown in figures 5 and 7) obviously derives from the timing of payments which now is considerably different. Given this is an incomplete markets setting the timing of cash flows is crucial and the decaying coupon bonds smooth the cash flow. When the government issues b_0 in $t = 0$ they have to pay b_0 in $t = 1$ using taxes and new debt, $b_0\delta_L + b_1$ in $t = 2$ and $b_0\delta_L^t + b_1\delta_L^{t-1} + \ldots + b_{t-1}$ in any generic t. This explains the mild hump shape response we see in the figures.

Note that there are significant differences between the allocations generated by the full commitment and the independent powers model, even when initial debt is zero (Figure 7). In the independent powers model the government has no incentive by construction to engage in interest rate twisting. The differences are due to the different set of state variables of each model. As we have seen earlier, in the commitment model the state vector includes the entire history of the λs, in the independent powers model taxes are a function of the inherited debt stock and the level of spending[26]. These results persist also when initial debt is positive (Figure 8). The effects of interest rate twisting a vaguely noticeable, the dominant force behind the tax response is the timing of payments. Figure 9 shows that these results survive when other maturities are considered.

[Tables 8 and 9 About Here.]

Now considering the short and long horizon simulations, Table 9 shows that taxes are more volatile under independent powers and the standard deviation increases with the horizon of the simulations. In short samples when the two models generate a similar (negative) average market value of debt, tax volatility of the independent power model is 50% higher than the one of the commitment model. At longer horizons and under independent powers the government accumulates a larger stock of precautionary savings. This brings the standard deviations of the tax rates closer to the commitment model. Nonetheless only part of the gap is closed: taxes are 20% more volatile in the independent power model.

B.4 Model Comparisons

We now compare the behaviour of the three models. We simulate all our models (buyback, decaying coupon, no buy back with coupons) under full commitment and independent power models, with the same 200 period shock sample and assuming b_0 to be 60% of GDP (the average of government debt in the US economy in the period 1955-2011).

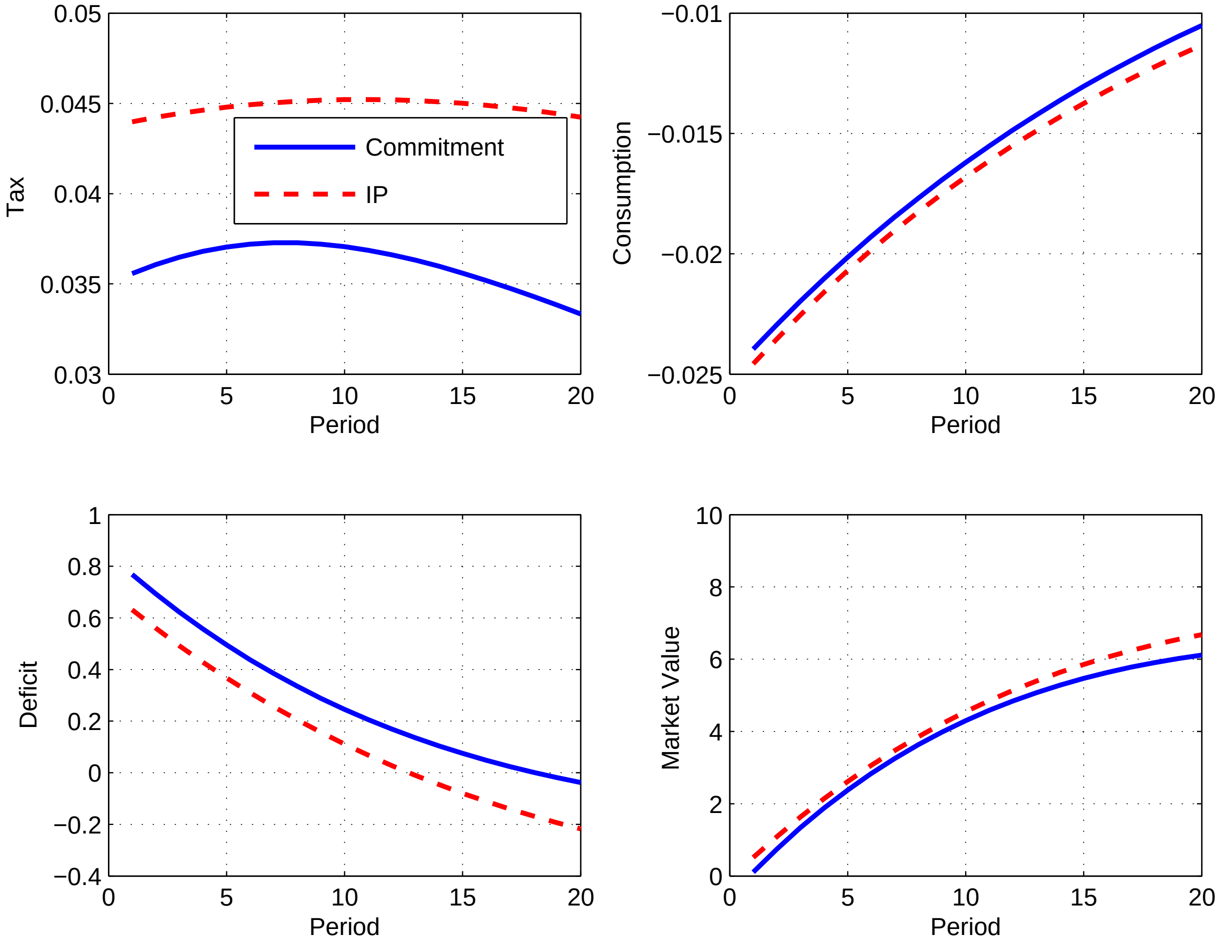
[26]Notice that this reasoning helps explain the differences in taxes in the first period across the two models, since these initial conditions influence one allocation but not the other.

In Figure 10 we show tax rates under commitment in the top panel and independent powers in the bottom one. The solid line represents the buyback model of Section 2, the dashed line shows taxes under constant coupons and no-buyback and the dashed dotted line shows the case of the decaying coupons model. In the case of commitment the tax schedules exhibit very similar behavior mirroring our previous results that under full commitment tax smoothing is a key policy objective. The variables which mainly influence taxes are the level of government spending and the level of debt. However, in the independent power model tax rates exhibit dramatically different behaviors due to the buyback assumption. The volatility of tax rates increases and the difference from the commitment models is much greater under no-buyback.

In Figure 11 we show the behavior of the market value of debt expressed as a percentage of GDP for the same shock sample that generated the tax rates. The evolution of the debt aggregate confirms the previous results. In the case of full commitment the three debt levels in the top panel track closely one another. But under no commitment the deviations are considerable. Under long bonds, changes in the institutional set up can generate substantially different debt behaviour.

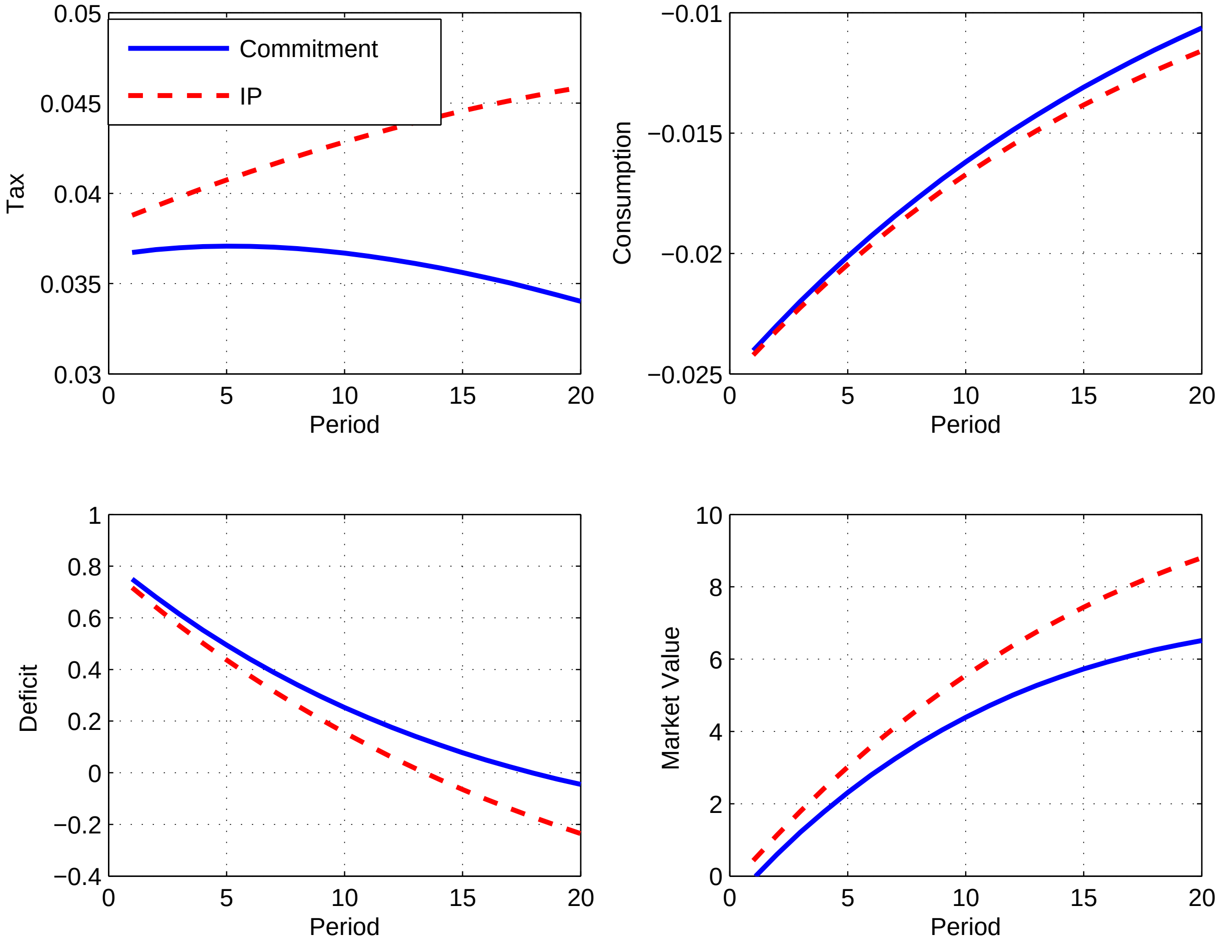
[Figures 10 and 11 About Here.]

Figure 7: **Responses under Zero Initial Debt: Decaying Coupons**



Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The starting value of debt is zero.

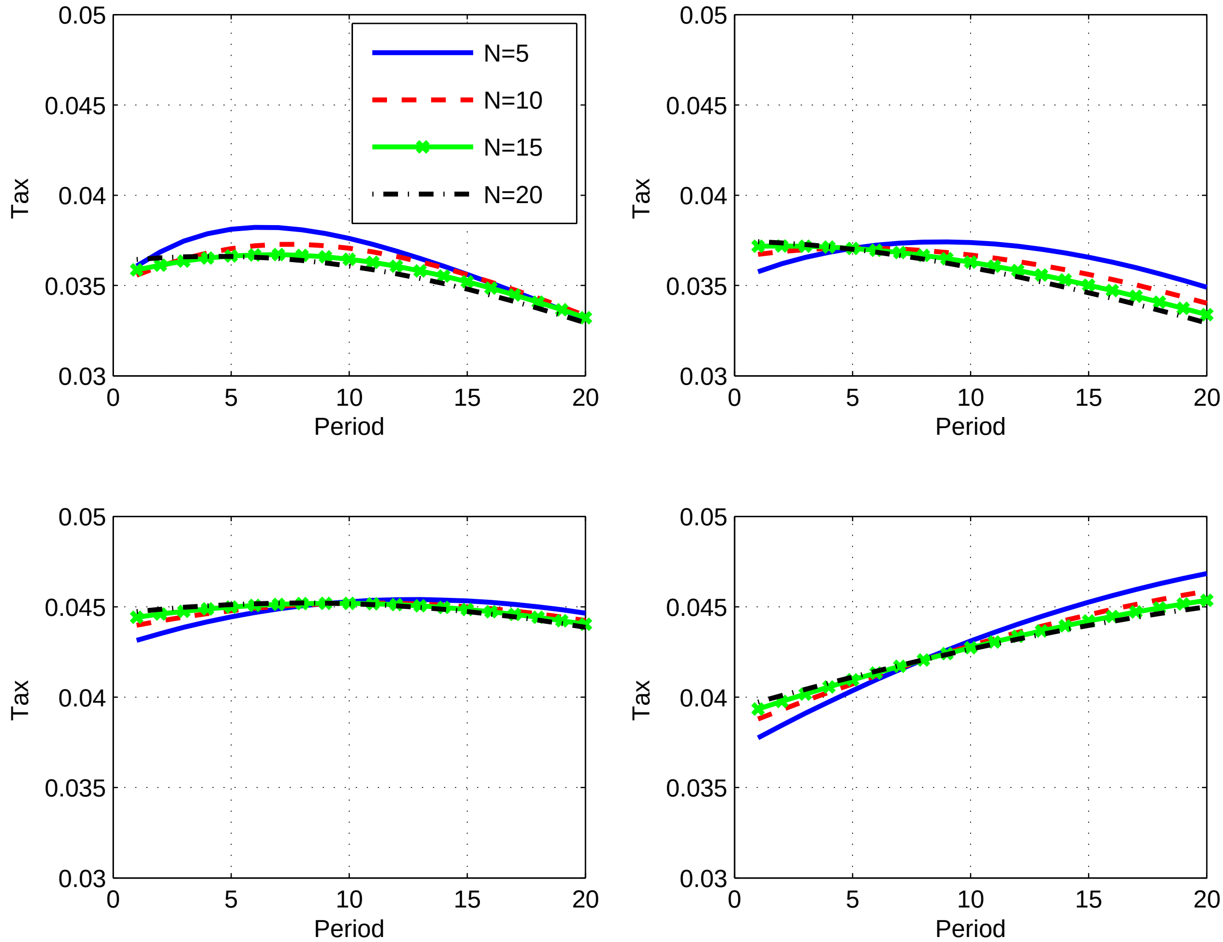
Figure 8: **Responses under Positive Initial Debt: Decaying Coupons**



Notes: The Figure plots the impulse response of taxes (top- left panel), consumption (top-right panel), deficit (bottom- left) and the market value of debt (bottom-right) to a one standard deviation shock in government spending. The maturity of long debt is $N = 10$ years. The quantities represented by the solid (blue) lines correspond to the optimal commitment allocation and the quantities plotted with dashed (red) lines correspond to the independent power model. The initial debt level is 50

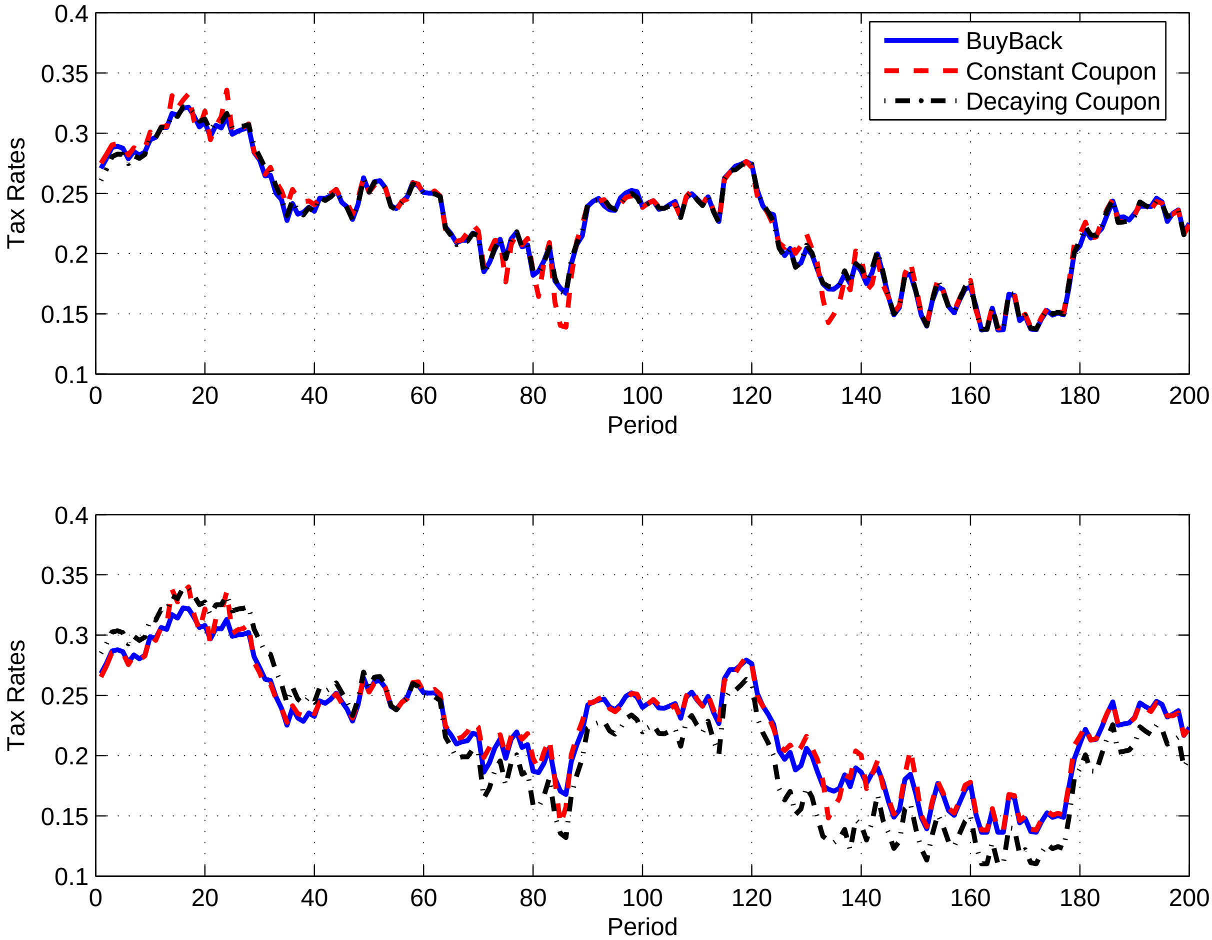
par

Figure 9: **Impulse Responses of Taxes: Decaying Coupons**



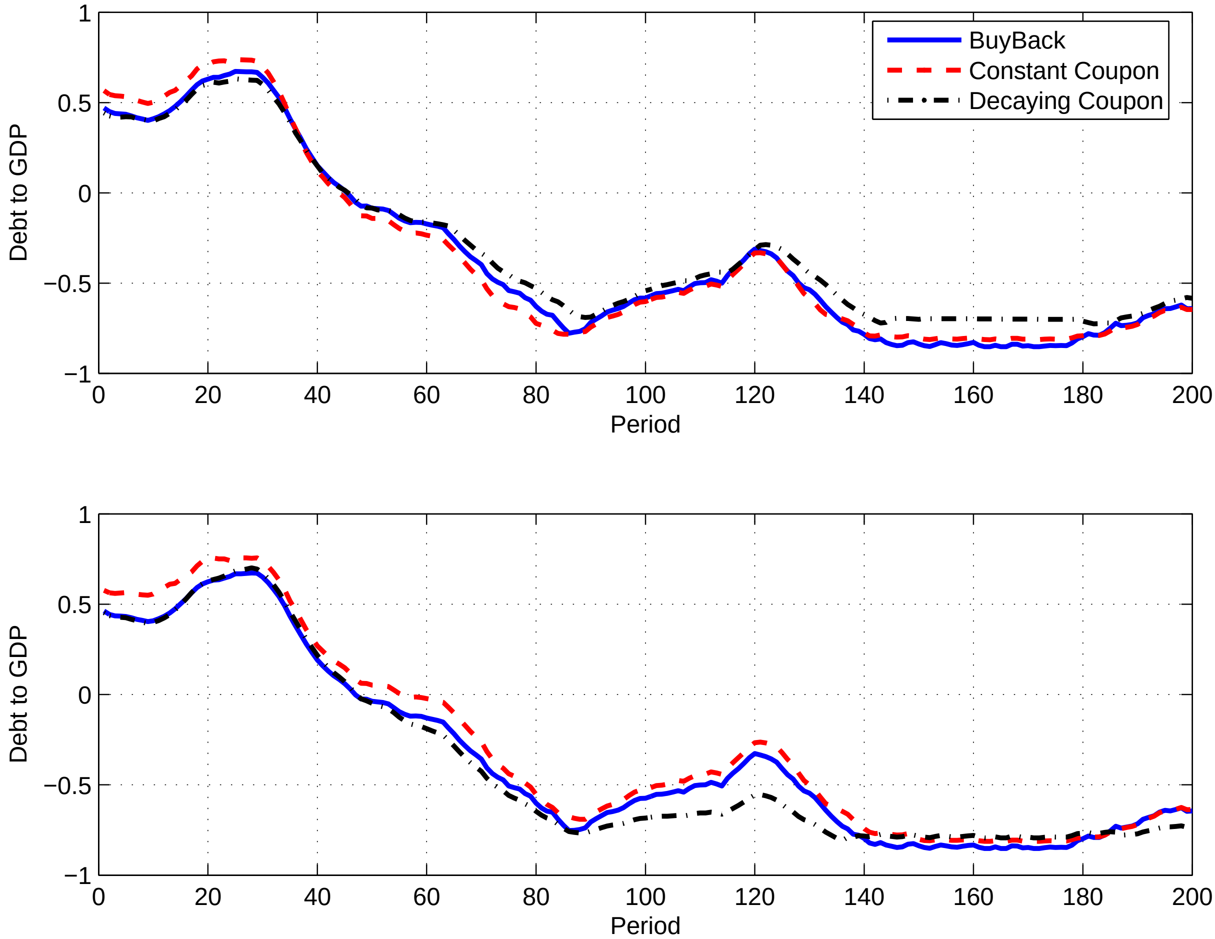
Notes: The Figure plots the impulse response of taxes to a spending shock. The top panels show the commitment model under decaying coupons with zero (left) and positive (right) debt levels. The solid line shows the response of the tax schedule when the maturity is $N = 5$. The dashed line corresponds to $N = 10$ and the crossed and dashed-dotted lines to $N = 15$ and $N = 20$ respectively. The bottom panels in the figure show the analogous responses in the case of independent powers.

Figure 10: **Tax Simulations: Various Models**



Notes: The Figure plots a simulated path for tax rates. The top panel show the case of the commitment model and the bottom panels the case of no commitment. We assume a starting value of government debt equal to 60 per cent of steady state GDP.

Figure 11: **Market Value of Debt Simulations: Various Models**



Notes: The Figure plots a simulated path for the market value of debt. The top panel show the case of the commitment model and the bottom panels the case of no commitment. We assume a starting value of government debt equal to 60 per cent of steady state GDP.

Table 8: **Moments: Commitment Model, Decaying Coupons**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.49	0.253	-0.214	-7.76	52.56	0.247	0.148	-14.26	52.57	0.246	0.274	-17.30
	10	52.49	0.253	-0.201	-7.03	52.56	0.248	0.119	-13.01	52.56	0.246	0.258	-16.50
	15	52.49	0.253	-0.194	-6.71	52.56	0.248	0.119	-11.77	52.56	0.246	0.253	-16.21
	20	52.50	0.253	-0.188	-6.54	52.56	0.248	0.115	-12.73	52.56	0.246	0.246	-15.98
std													
	5	2.27	0.0220	1.18	10.47	3.13	0.0356	1.54	21.12	3.48	0.0435	1.59	29.70
	10	2.28	0.0220	1.16	10.24	3.13	0.0356	1.52	21.33	3.48	0.0436	1.58	30.16
	15	2.28	0.0220	1.14	10.05	3.14	0.0357	1.51	21.36	3.48	0.0437	1.58	30.41
	20	2.28	0.0220	1.14	9.94	3.14	0.0357	1.51	21.33	3.48	0.0438	1.58	30.63

Notes: The table shows key moments from the commitment model with decaying coupons. The initial conditions for government debt are uniformly distributed in the interval $[-\overline{M}, \overline{M}]$.

Table 9: **Moments: Independent Powers, Decaying Coupons**

		t=40				t=200				long run			
	N	c	τ	$deficit$	MV	c	τ	$deficit$	MV	c	τ	$deficit$	MV
mean													
	5	52.53	0.249	0.049	-12.12	52.78	0.233	1.13	-26.36	52.88	0.225	1.696	-34.97
	10	52.53	0.249	0.039	-11.15	52.76	0.234	1.05	-24.53	52.86	0.226	1.591	-32.92
	15	52.53	0.249	0.035	-10.61	52.75	0.235	1.00	-23.51	52.84	0.227	1.530	-31.69
	20	52.53	0.249	0.032	-10.27	52.74	0.235	0.98	-22.88	52.83	0.228	1.494	-30.98
std													
	5	2.32	0.0266	1.19	10.54	3.23	0.0445	1.69	20.43	3.59	0.0533	1.73	25.91
	10	2.32	0.0265	1.16	10.30	3.23	0.0445	1.70	20.80	3.60	0.0542	1.80	27.44
	15	2.32	0.0265	1.15	10.18	3.23	0.0445	1.70	20.97	3.61	0.0546	1.84	28.22
	20	2.32	0.0265	1.14	10.10	3.23	0.0445	1.70	21.07	3.61	0.0549	1.86	28.68

Notes: The table shows key moments from the independent power model with decaying coupons. The samples used to construct the moments are the ones we used in Table 8.