

Stock Price Booms and Expected Capital Gains[†]

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Investors' subjective capital gains expectations are a key element explaining stock price fluctuations. Survey measures of these expectations display excessive optimism (pessimism) at market peaks (troughs). We formally reject the hypothesis that this is compatible with rational expectations. We then incorporate subjective price beliefs with such properties into a standard asset-pricing model with rational agents (internal rationality). The model gives rise to boom-bust cycles that temporarily delink stock prices from fundamentals and quantitatively replicates many asset-pricing moments. In particular, it matches the observed strong positive correlation between the price dividend ratio and survey return expectations, which cannot be matched by rational expectations. (JEL D83, D84, G12, G14)

Bull-markets are born on pessimism, grow on skepticism, mature on optimism and die on euphoria.

—Sir John Templeton, Founder of Templeton Mutual Funds

Following the recent boom and bust cycles in a number of asset markets around the globe, there exists renewed interest in understanding better the forces contributing

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to the emergence of such drastic asset price movements. This paper argues that movements in investor optimism and pessimism, as measured by the movements in investors' subjective expectations about future capital gains, are a crucial ingredient for understanding these fluctuations.

We present an asset-pricing model that incorporates endogenous belief dynamics about expected capital gains. The model gives rise to sustained stock price booms and busts and is consistent with the behavior of investors' capital gains expectations, as measured by survey data. The presented modeling approach differs notably from the standard approach in the consumption-based asset-pricing literature, which proceeds by assuming that stock price fluctuations are fully efficient. Campbell and Cochrane (1999) and Bansal and Yaron (2004), for example, present models in which stock price fluctuations reflect the interaction of investor preferences and stochastic driving forces in a setting with optimizing investors who hold rational expectations.

We first present empirical evidence casting considerable doubt on the prevailing view that stock price fluctuations are efficient. Specifically, we show that the rational expectations (RE) hypothesis gives rise to an important counterfactual prediction for the behavior of investors' return or capital gain expectations.¹ This counterfactual prediction is a model-independent implication of the RE hypothesis, but, as we explain below, key for understanding stock price volatility and its efficiency properties.

As previously noted by Fama and French (1988), the empirical behavior of asset prices implies that rational return expectations correlate negatively with the price dividend (*PD*) ratio. Somewhat counterintuitively, the RE hypothesis thus predicts that investors should be particularly pessimistic about future stock returns in the early part of the year 2000, when the tech stock boom and the *PD* ratio of the S&P 500 reached their all-time maximum. As we document, the available survey evidence implies precisely the opposite: all quantitative survey measures of investors' expected return (or capital gain) available for the US economy, unambiguously and unanimously, correlate positively with the *PD* ratio. And perhaps not surprisingly, return expectations reached a temporary maximum rather than a minimum in the early part of the year 2000, i.e., precisely at the peak of the tech stock boom, a fact previously shown in Vissing-Jorgensen (2004).

We present formal econometric tests of the null hypothesis that the survey evidence is consistent with RE and demonstrate that the hypothesis of rational return or capital gain expectations is overwhelmingly rejected by the data. Our tests correct for small sample bias, account for autocorrelations in the error structure, and are immune to the presence of differential information on the part of agents and to the presence of measurement error in survey data. An appealing feature of the tests is that they also provide clues about why the RE hypothesis fails: the failure arises because survey expectations and RE covary differently with the *PD* ratio, a finding that is useful for guiding the search for alternative and empirically more plausible expectations models.

The positive comovement of stock prices and survey expectations suggests that price fluctuations are amplified by overly optimistic beliefs at market peaks and by

¹ Since most variation in returns is due to the variation in capital gains, we tend to use both terms interchangeably.

overly pessimistic beliefs at market troughs. Furthermore, it suggests that investors' capital gains expectations are influenced, at least partly, by the capital gains observed in the past, in line with evidence presented by Malmendier and Nagel (2011). Indeed, a simple adaptive updating equation captures the time series behavior of the survey data and its correlation with the *PD* ratio very well.

Taken together, these observations motivate the construction of an asset-pricing model in which agents hold subjective beliefs about price outcomes. We do so using the framework of internal rationality (IR), developed in Adam and Marcet (2011), which allows us to consider maximizing investors that hold subjective price beliefs within an otherwise standard Lucas (1978) asset-pricing model.² Within this framework, agents optimally update beliefs using Bayes' law.

With this modification, the Lucas model becomes *quantitatively* consistent with important aspects of the data. Using confidence intervals based on the simulated method of moments (SMM), we find that the model matches key moments describing the observed volatility of stock prices and the positive correlation between the *PD* ratio and subjective return expectations. This is obtained even though we use the simplest version of the Lucas model with time separable preferences and standard stochastic driving processes. The same model under RE is very far from explaining the data and produces, amongst other things, far too little price volatility and the wrong sign for the correlation between the *PD* ratio and investors return expectations.

The strong improvement in the model's empirical performance arises because agents' attempts to improve their knowledge about price behavior can temporarily delink asset prices from their fundamental (RE) value and give rise to belief-driven boom and bust cycles in stock prices. This occurs because with imperfect information about the price process, optimal behavior prescribes that agents use past capital gain observations to learn about the stochastic process governing the behavior of capital gains; this generates a feedback between capital gain expectations and realized capital gains that can drive booms and busts in stock prices.

Suppose—in line with the empirical evidence—that agents become more optimistic about future capital gains whenever they are positively surprised by past capital gains.³ A positive surprise for the capital gains observed in the previous period then increases optimism about the capital gains associated with investing in the asset today. If such increased optimism leads to an increase in investors' asset demand and if this demand effect is sufficiently strong, then positive past surprises trigger further positive surprises today, and thus further increases in optimism tomorrow. As we show analytically, stock prices in our model do increase with capital gain optimism whenever the substitution effect of increased optimism dominates the wealth effect of such belief changes. Asset prices in the model then display sustained price booms, similar to those observed in the data.

After a sequence of sustained increases, countervailing forces come into play that endogenously dampen the upward price momentum, eventually halt it, and cause a reversal. Specifically, in a situation where increased optimism about capital gains

² As is explained in Adam and Marcet (2011), subjective price beliefs are consistent with optimizing behavior in the presence of lack of common knowledge about agents' beliefs and preferences.

³ Such positive surprises may be triggered by fundamental shocks, e.g., a high value for realized dividend growth.

has led to a stock price boom, stock prices make up for a larger share of agents' total wealth.⁴ As we show analytically, this causes the wealth effect to become as strong as (or even stronger than) the substitution effect when expectations about stock price appreciation are sufficiently high.⁵ Increases in optimism then cease to cause further increases in stock demand and thus stock prices, so that investors' capital gains expectations turn out to be too optimistic relative to the realized outcomes. This induces downward revision in beliefs, which gives rise to negative price momentum and an asset price bust.

The previous arguments show how belief dynamics can temporarily delink asset prices from their fundamental value. Clearly, these price dynamics are inefficient as they are not justified by innovations to preferences or other fundamentals.

Since we depart from RE, our model requires introducing an explicit assumption about agents' price beliefs. Various elements guide this modeling choice. First, we choose price beliefs such that there are no *black swan*-like events, i.e., we insure that agents have contingency plans for all prices that they actually encounter along the equilibrium path. Second, we choose the subjective price process such that it gives rise to capital gain expectations that are consistent with the behavior of survey expectations. In particular, agents believe the average growth rate of stock prices to slowly drift over time, which is consistent with the presence of prolonged periods of price booms that are followed by price busts. Given these beliefs, equilibrium prices will indeed display prolonged periods of above average and below average growth.

More generally, the present paper shows how the framework of internal rationality allows studying learning about market behavior in a model of intertemporal decision making. It thereby improves on shortcomings present in the learning literature, where agents' belief-updating equations and choices are often not derived from individual maximization and are optimal only in the limit once learning converges to the RE outcome. We thus provide explicit microfoundations for models of adaptive learning about market outcomes.

The bulk of the paper considers a representative agent model. This is motivated by the desire to derive results analytically and to show how a rather small deviation from the standard paradigm helps reconciling the model with the data. A range of extensions consider, amongst other things, a heterogeneous agents version and more elaborate subjective belief structures. These extensions allow replicating additional data features, e.g., the equity premium.

The remainder of the paper is structured as follows. The next section discusses the related literature. Section II then shows that the price dividend ratio (*PD*) ratio covaries positively with survey measures of investors' return expectations and that this is incompatible with the RE hypothesis. It also shows that the time series of survey expectations can be captured by a fairly simple belief updating equation. Section III introduces our asset-pricing model with subjective beliefs. As a benchmark, Section IV determines the RE equilibrium. Section V introduces a specific model of subjective price beliefs, which relaxes agents' prior about price

⁴This occurs because stock prices are high, but also because agents discount other income streams, e.g., wage income, at a higher rate.

⁵With constant relative risk aversion (CRRA) utility, this happens whenever the coefficient of relative risk aversion is larger than 1.

behavior relative to the RE equilibrium beliefs. It also derives the Bayesian updating equations characterizing the evolution of subjective beliefs over time. After imposing market clearing in Section VI, we present closed-form solutions for the *PD* ratio in Section VII in the special case of vanishing uncertainty. Using the analytical solution, we explain how the interaction between belief updating dynamics and price outcomes can endogenously generate boom and bust dynamics in asset prices. Section VIII estimates the fully stochastic version of the model using a mix of calibration and simulated method of moments estimation. It shows that the model successfully replicates a number of important asset-pricing moments, including the positive correlation between expected returns and the *PD* ratio. It also explains how the model gives rise to a high Sharpe Ratio and to a low volatility for the risk-free interest rate. Section IX shows that the estimated model can replicate the low frequency movements in the time series of the US postwar *PD* ratio, as well as the available time series of survey data. Section X presents a number of robustness checks and extensions of the basic model. A conclusion briefly summarizes and discusses potential avenues for future research. Technical material and proofs can be found in the online Appendix.

I. Related Literature

Following Shiller's (1981) seminal observation that stock price volatility cannot be explained by the volatility of rational dividend expectations, the asset-pricing literature made considerable progress in explaining stock price behavior. Bansal and Yaron (2004) and Campbell and Cochrane (1999), for example, developed consumption-based RE models in which price fluctuations result from large and persistent swings in investors' stochastic discount factor. Section II shows, however, that RE models fail to capture the behavior of investors' return expectations. This strongly suggests that RE models fall short of providing a complete explanation of the sources of stock price volatility.

Attributing stock price fluctuations to *sentiment* fluctuations or issues of learning has long had an intuitive appeal. A substantial part of the asset-pricing literature introduces subjective beliefs to model investor *sentiment*. The standard approach resorts to Bayesian RE modeling, which allows for subjective beliefs about fundamentals, while keeping the assumption that investors know the equilibrium pricing function linking stock prices to fundamentals. Following early work by Timmermann (1993) and Barberis, Shleifer, and Vishny (1998), a substantial literature follows this approach. It finds that the additional stock price volatility generated from learning is overall small compared to the gap that exists relative to the data.

Recent work by Barberis et al. (2015), for example, considers a time-separable utility framework where some investors have rational dividend beliefs while others extrapolate from past dividend observations.⁶ While this allows one to successfully

⁶Barberis et al. (2015, p. 4) claim that agents agree on dividend behavior and that the only difference between rational agents and extrapolators is that the latter extrapolate future prices from past prices. Yet, as it turns out, their model is one where some agents extrapolate past dividends while others have rational dividend expectations and all agents know the equilibrium pricing function, in line with standard Bayesian RE modeling. This can be seen in the proof of their Proposition 1, which reverse-engineers a process for extrapolators' dividend beliefs (ibid., equations (A16) and (A18)) that differs from the true dividend process (ibid., equation (1)). The proof of the proposition

replicate survey behavior, the standard deviation of the *PD* ratio falls one order of magnitude short of that observed in the data, so that there are no significant stock price boom and bust episodes.⁷

In ongoing work, Hirshleifer, Li, and Yu (2015) and Choi and Mertens (2013) consider Bayesian RE models with time-nonseparable preferences and investors who extrapolate past fundamentals. They show how extrapolation of fundamentals endogenously generates long-run consumption risk and thereby some increase in asset price volatility. Collin-Dufresne, Johannes, and Lochstoer (2016) show how learning about fundamentals can give rise to sizable Sharpe ratios, provided agents have a preference for an early resolution of uncertainty. Standard preference parameterizations in this class of models imply, however, that agents are willing to give up a big part of consumption to resolve consumption risk early (see Epstein, Farhi, and Strzalecki 2014).

The modeling approach pursued in the present paper differs fundamentally from the one discussed in the previous paragraphs. The Bayesian RE literature assumes that agents find it difficult to forecast fundamental shocks (agents hold subjective dividend beliefs), but that agents can predict perfectly price outcomes conditional on the history of observed fundamentals (agents know the equilibrium pricing function mapping the history of dividends into price outcomes). Assuming that agents know the pricing function provides agents with a substantial amount of information about market behavior, suggesting that it is of interest to study the effects of relaxing this informational assumption as we do in the current paper. Our agents entertain a distribution of prices for given fundamentals, which is nondegenerate and that does not coincide with the model at hand.⁸

To show that a key element for understanding stock price volatility is investor's imperfect knowledge about how prices are formed, we make the distinction to the Bayesian RE literature as stark as possible: we assume that agents find it easy to predict fundamentals, i.e., assume agents hold RE about dividends, but find it difficult to predict price behavior, i.e., agents do not know the equilibrium pricing function. We show that a simple asset-pricing model can then replicate survey data and generate sufficient volatility for the *PD* ratio, including occasional boom and bust episodes. This is achieved in a setting with standard time-separable preferences and obtained because there is a much stronger propagation of economic disturbances when agents learn about the equilibrium pricing function: belief changes then affect stock price behavior and stock prices feed back into belief changes. This allows movements in prices and beliefs to mutually reinforce each other during price boom

explicitly acknowledges that these subjective dividend beliefs are chosen such that extrapolators' price beliefs (ibid., equations (3) and (4)) are consistent with the equilibrium pricing function: "At the same time, in order to compute the values of the derived parameters that govern their consumption and portfolio decisions, extrapolators need to be aware of the price equation (A11)" (ibid., p. 19). This implies that rational agents and extrapolators disagree about the dividend process but agree about the equilibrium pricing function, in line with standard Bayesian RE modeling. It also implies that in their setting, learning from price behavior is observationally equivalent to learning from dividend behavior, unlike in the setting presented in this paper.

⁷Due to the constant absolute risk aversion (CARA) utility setup, the volatility of the *PD* ratio also asymptotically converges to zero.

⁸This is related to work by Angeletos and La'O (2013), who consider a setting in which agents are uncertain about the price at which they will be able to trade. They show how sentiment shocks can give rise to perfectly self-fulfilling fluctuations in aggregate outcomes. Sentiment shocks in their setting result from extrinsic uncertainty; in our setting they are triggered from intrinsic sources of uncertainty.

and bust phases, thereby increasing price volatility. The feedback from market outcomes into beliefs is absent in a Bayesian RE setting.

The literature on robust control and asset prices (e.g., Cogley and Sargent 2008) considers settings where investors are uncertain about the process for fundamentals. In line with Bayesian RE modeling, this literature assumes that investors know the equilibrium pricing function.

The literature on adaptive learning previously considered deviations from rational price expectations using asset-pricing models where investors learn about price behavior. Marcet and Sargent (1992), for example, study convergence to RE when agents estimate an incorrect model of stock prices by least squares learning. A range of papers in the adaptive learning literature argues that learning generates additional stock price volatility. Bullard and Duffy (2001) and Brock and Hommes (1998), for example, show that learning dynamics can converge to complicated attractors that increase asset return volatility, when the RE equilibrium is unstable.⁹ Lansing (2010) shows how near-rational bubbles can arise in a model with learning about price behavior. Branch and Evans (2011) present a model where agents learn about risk and return and show how it gives rise to bubbles and crashes. Boswijk, Hommes, and Manzan (2007) estimate a model with fundamentalist and chartist traders whose relative shares evolve according to an evolutionary performance criterion, showing that the model can generate a run-up in asset prices and subsequent mean-reversion to fundamental values. De Long et al. (1990) show how the pricing effects of positive feedback trading survives or even get amplified by the introduction of rational speculators.

The approach used in the present paper differs along several dimensions from the contributions mentioned in the previous paragraph. First, we compare quantitatively the implications of our model with the data, i.e., we match a standard set of asset-pricing moments capturing stock price volatility and use formal asymptotic distribution to evaluate the goodness of fit. Second, we compare the model to evidence obtained from survey data. Third, we present a model that derives investors' consumption and stockholding plans from properly specified microfoundations. In particular, we consider agents who solve an infinite horizon decision problem and hold a consistent set of beliefs, and we discuss conditions for existence and uniqueness of optimal plans as well as conditions insuring that the optimal plan has a recursive representation. The adaptive learning literature often relies on shortcuts that amount to introducing additional behavioral elements into decision making and postulates beliefs that become well specified only in the limit, if convergence to RE occurs.¹⁰

In prior work, Adam, Marcet, and Nicolini (2016) present a model in which investors learn about risk-adjusted price growth and show how such a model can quantitatively replicate a set of standard asset-pricing moments describing stock price volatility. While replicating stock price volatility and postulating beliefs that are hard to reject in the light of the existing asset price data and the outcomes generated by the model, their setup falls short of explaining survey evidence. Specifically, it counterfactually implies that stock return expectations are constant over time. Adam, Marcet, and Nicolini (2016) also solve for equilibrium prices under the assumption that dividend and trading income are a negligible part of total income.

⁹Stability under learning dynamics is defined in Marcet and Sargent (1989).

¹⁰See Adam and Marcet (2011, Section 2) for a detailed discussion.



FIGURE 1. QUARTERLY PD RATIO OF THE S&P 500, 1946:1–2012:1

Notes: PD ratio is defined as stock prices over quarterly dividends. Dividends are deseasonalized by averaging dividend payments over the current and previous three quarters.

We solve the model without this assumption and show that it can play an important role for the model solution, for example, it gives rise to an endogenous upper bound for equilibrium prices.¹¹

The experimental and behavioral literature provides further evidence supporting the presence of subjective price beliefs. Hirota and Sunder (2007) and Asparouhova et al. (2016), for example, implement the Lucas asset-pricing model in the experimental laboratory and document that there is excess volatility in prices that is unaccounted for by the rational expectations equilibrium and that likely arises from participants' expectations about future prices. Furthermore, the type of learning employed in the present model is in line with evidence presented in Malmendier and Nagel (2011), who show that experienced returns affect beliefs about future asset returns.¹²

II. Stock Prices and Stock Price Expectations

This section shows that survey expectations of future stock prices are inconsistent with the notion that agents hold rational stock price expectations. Indeed, our formal econometric tests, presented in Sections IIB and IIC, show that stock market investors display undue optimism about future stock prices when the *PD* ratio is high and undue pessimism when the *PD* ratio is low. Section IID then illustrates how simple adaptive price predictions quantitatively capture the relationship between survey expectations and the *PD* ratio.

¹¹ In line with the approach in the Bayesian RE literature, Adam, Marcet, and Nicolini (2016) impose an exogenous upper bound on agents' beliefs, a so-called *projection facility*, so as to insure existence of finite equilibrium prices.

¹² Greenwood and Nagel (2009) show that—in line with this hypothesis—young mutual fund managers displayed trend-chasing behavior over the tech stock boom and bust around the year 2000.

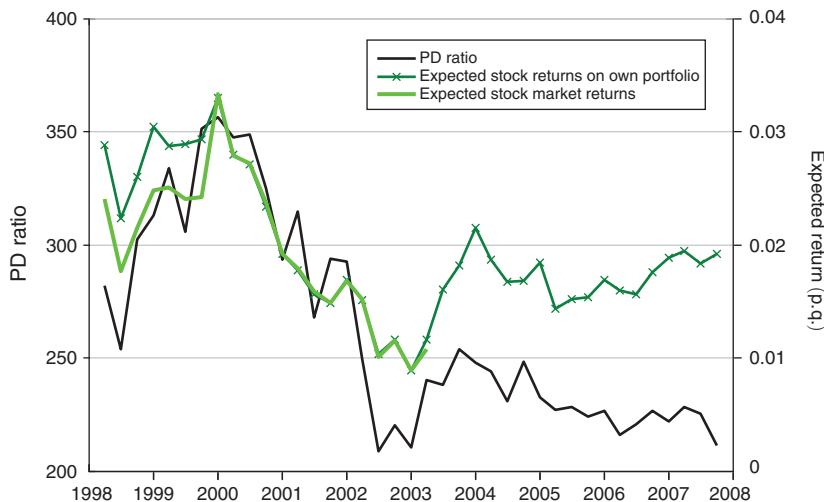


FIGURE 2. QUARTERLY PD RATIO OF THE S&P 500 (*LHS scale*) AND INVESTORS' EXPECTED QUARTERLY REAL RETURNS FROM THE UBS GALLUP SURVEY (*RHS scale*), 1998:II–2007:IV

A. Survey Expectations and the PD Ratio

As pointed out by Vissing-Jorgensen (2004), Greenwood and Shleifer (2014), and Adam and Marcet (2010), survey expectations of future returns (or capital gains) display a positive correlation with the *PD* ratio, while actual returns (or capital gains) display a negative correlation.¹³

Online Appendix A2 documents this fact for a range of surveys and Figure 2 illustrates it using our preferred survey, the UBS Gallup Survey, which is based on a representative sample of approximately 1,000 US investors who own at least US\$10,000 in financial wealth.¹⁴ Figure 2 graphs the US *PD* ratio (the black line) together with measures of the cross-sectional average of investors' one-year ahead expected real return.¹⁵ Return expectations are expressed in terms of quarterly real growth rates and the figure depicts two expectations measures: investors' expectations about the one year ahead stock market return, as well as their expectations about the one year ahead returns on their own stock portfolio. These measures behave very similarly over the period for which they overlap, but the latter is available for a longer time period. Figure 2 reveals that there is a strong positive correlation between the *PD* ratio and expected returns. The correlation between the expected own portfolio returns and the *PD* ratio is +0.70 and even higher for expected stock market returns (+0.79). Moreover, investors' return expectations were highest at the beginning of the year 2000, which is precisely the year the *PD* ratio reached its

¹³ A related observation is that return forecast errors implied by survey data can be predicted using the *PD* ratio: see Bacchetta, Mertens, and Van Wincoop (2009).

¹⁴ About 40 percent of respondents own more than US\$100,000 in financial wealth. As is documented in online Appendix A2, this subgroup does not behave differently.

¹⁵ To be consistent with the asset-pricing model presented in later sections, we report expectations of real returns. The nominal return expectations from the survey have been transformed into real returns using inflation forecasts from the Survey of Professional Forecasters. Results are robust to using instead the Michigan Survey inflation forecast.

peak during the tech stock boom. Investors then expected annualized real returns of around 13 percent from stock investments, while the subsequently realized returns turned out to be particularly dismal. Conversely, investors were most pessimistic in the year 2003 when the *PD* ratio reached its bottom, expecting then annualized real returns of below 4 percent.

This evidence suggests that survey data are incompatible with rational expectations and that stock prices seem to play a role in the formation of expectations about stock returns. Yet, evidence based on comparing two correlations can only be suggestive, as it is subject to several econometric shortcomings. For example, if investors possess information that is not observed by the econometrician, as might be considered likely, then the correlation between the fully rational return forecasts and the *PD* ratio will differ from the correlation between realized returns and the *PD* ratio. The same holds true if survey expectations are measured with error, as one can reasonably expect. Furthermore, results in Stambaugh (1999) imply that with the *PD* ratio being such a persistent process, there is considerable small sample bias in these correlations, given the relatively short time spans over which investor expectations can be tracked. Finally, a highly serially correlated predictor variable (*PD* ratio), whose innovations are correlated with the variable that is to be predicted (future returns), gives rise to spurious regression and thus spurious correlation problems (see Ferson, Sarkissian, and Simin 2003 and Campbell and Yogo 2006). There also does not exist a standard nonparametric approach allowing to correct for these small sample issues when comparing correlations.¹⁶ Comparisons involving correlations are thus insufficient for rejecting the hypothesis that survey expectations are consistent with RE. To deal with these concerns, the next sections construct formal econometric tests that take these concerns fully into account.

B. RE Test with Small Sample Adjustments

This section develops a RE test that takes into account the concerns expressed in the previous section. While the present section emphasizes the derivation of analytical results, Section IIC provides further tests that rely entirely on Monte Carlo simulation.

Let $E_t^{\mathcal{P}}$ denote agents' subjective expectations operator based on information up to time t , which can differ from the rational expectations operator E_t . Let $R_{t,t+N}$ denote the real cumulative stock returns between period t and $t+N$ and let $\mathcal{E}_t^N = E_t^{\mathcal{P}} R_{t,t+N} + \mu_t^N$ denote the (potentially noisy) observation of expected returns, as obtained, for example, from survey data, where μ_t^N is measurement error.¹⁷

Let us linearly project the random variable $E_t^{\mathcal{P}} R_{t,t+N}$ on $\frac{P_t}{D_t}$ to define

$$(1) \quad E_t^{\mathcal{P}} R_{t,t+N} = a^N + c^N \frac{P_t}{D_t} + u_t^N,$$

¹⁶ Any test must take into account the joint distribution of the correlation estimates in order to make statistically valid statements.

¹⁷ Since the Shiller survey reports expectations about capital gains instead of returns, $R_{t,t+N}$ denotes the real growth rate of stock prices between periods t and $t+N$ when using the Shiller survey.

where

$$(2) \quad E(x_t u_t^N) = 0,$$

for $x_t' = (1, P_t/D_t)$. The operator E denotes the objective expectation for the *true* data-generating process, whatever is the process for agents' expectations. The projection residual u_t^N captures variations in agents' actual expectations that cannot be linearly attributed to the price-dividend ratio.¹⁸ It summarizes all other information that agents believe to be useful in predicting $R_{t,t+N}$.¹⁹

Due to the potential presence of measurement error, one cannot directly estimate equation (1), but given the observed return expectations \mathcal{E}_t^N , one can write the following *regression* equation:

$$(3) \quad \mathcal{E}_t^N = a^N + c^N \frac{P_t}{D_t} + u_t^N + \mu_t^N.$$

Assuming that the measurement error μ_t^N is orthogonal to the current *PD* ratio,²⁰ we have the orthogonality condition

$$(4) \quad E[x_t(u_t^N + \mu_t^N)] = 0.$$

Let \hat{c}_T^N denote the ordinary least squares (OLS) estimator of c^N in equation (3), given a sample of length T .²¹

We can specify an additional *regression* equation like equation (3), but with actual returns as dependent variable,

$$(5) \quad R_{t,t+N} = \mathbf{a}^N + \mathbf{c}^N \frac{P_t}{D_t} + \mathbf{u}_t^N,$$

where

$$(6) \quad E[x_t \mathbf{u}_t^N] = 0.$$

Let $\hat{\mathbf{c}}_T^N$ denote the OLS estimate of \mathbf{c}^N with T observations.

The reader can probably guess that the regression estimates are useful here because under the hypothesis of RE we have $c^N = \mathbf{c}^N$, so that the estimates \hat{c}_T^N and $\hat{\mathbf{c}}_T^N$ are both consistent estimates of the *same* parameter, letting the prediction error $\varepsilon_t^N = R_{t,t+N} - E_t(R_{t,t+N})$, where E_t is taken with respect to the objective distribution and the information available to investors at t , we have $\mathbf{u}^N = u^N + \varepsilon_t^N$. This gives rise to the following test.

¹⁸The residual u_t^N is likely to be correlated with current and past observables (other than the *PD* ratio) and thus serially correlated.

¹⁹The projection in equation (1) and the error are well defined as long as agents' expectations $E_t^P R_{t,t+N}$ and P_t/D_t are stationary and have bounded second moments.

²⁰We allow μ_t^N to be serially correlated and correlated with equilibrium variables other than *PD*.

²¹Although the residuals u_t^N and the measurement errors μ_t^N are likely to be serially correlated, the OLS estimate is consistent.

PROPOSITION 1: Assume the process $\{R_{t,t+N}, \mathcal{E}_t^N, P_t/D_t\}$ is stationary and ergodic, all moments are such that asymptotic distributions exist,²² $E(\mu_t) = E(\mu_t P_t/D_t) = 0$, and P_t/D_t is part of agents' time t information set. Then

(i) Under the null hypothesis of rational expectations

$$(7) \quad \sqrt{T} \frac{\hat{\mathbf{c}}_T^N - \hat{\mathbf{c}}_T^N}{\hat{\sigma}_{c-c}} \rightarrow N(0, 1) \text{ in distribution as } T \rightarrow \infty,$$

where $\hat{\sigma}_{c-c}^2$ is a consistent estimate of $\text{var}(\hat{\mathbf{c}}^N - \hat{\mathbf{c}}^N)$.²³

(ii) Suppose in addition that

$$(8) \quad \frac{P_t}{D_t} = \overline{PD}(1 - \rho) + \rho \frac{P_{t-1}}{D_{t-1}} + \varepsilon_t^{PD}$$

for $\rho \in (-1, 1)$, where $(u_t^N + \mu_t^N, \mathbf{u}_t^N, \varepsilon_{t+1}^{PD})$ is normally distributed, i.i.d., with mean 0, and $E(\mu_t^N \varepsilon_{t+1}^{PD}) = 0$. Under the null hypothesis of rational expectations, the small sample bias of $\hat{\mathbf{c}}_T^N - \hat{\mathbf{c}}_T^N$ in the test-statistic (7) is

$$(9) \quad E(\hat{\mathbf{c}}_T^N - \hat{\mathbf{c}}_T^N) = \frac{\text{cov}(\varepsilon_{t+1}^{PD}, \varepsilon_t^N)}{\text{var}(\varepsilon_t^{PD})} E(\hat{\rho}_T - \rho),$$

where $E(\hat{\rho}_T - \rho)$ is the small sample bias in the estimation of ρ for a sample of length T .

The proof of Proposition 1 can be found in online Appendix A3. It treats equations (3)–(6) as a seemingly unrelated regression system and uses the fact that under rational expectations one has $\mathbf{c}^N = \mathbf{c}^N$. Part (ii) of the proposition follows from results in Stambaugh (1999).

Part (i) of Proposition 1 uses minimal assumptions to obtain an asymptotically valid result. Essentially, all that is needed is stationarity of the observables and orthogonality of the measurement error. The test is asymptotically robust to serial correlation and heteroskedasticity of the error terms.

Part (ii) of Proposition 1 deals with small sample bias in the test statistic. Since ρ is close to 1, we have $E(\hat{\rho}_T - \rho) < 0$; and since future stock price increases are likely to be correlated with future surprises to returns, i.e., $\text{cov}(\varepsilon_{t+1}^{PD}, \varepsilon_t^N) > 0$, we tend to get $E(\hat{\mathbf{c}}_T^N - \hat{\mathbf{c}}_T^N) < 0$ in small samples, even if in fact $\mathbf{c}^N = \mathbf{c}^N$.

²²More precisely, we assume: (i) bounded second moments of $(R_{t,t+N}, \mathcal{E}_t^N, P_t/D_t)$; (ii) $\text{var}(P_t/D_t) > 0$; and

$$S_w = \sum_{k=-\infty}^{\infty} E \left(\begin{bmatrix} u_t^N + \mu_t^N \\ u_t^N + \varepsilon_t^N \end{bmatrix} \begin{bmatrix} u_{t-k}^N + \mu_{t-k}^N \\ u_{t-k}^N + \varepsilon_{t-k}^N \end{bmatrix}' \otimes x_t x_{t-k}' \right) < \infty.$$

²³Equation (A5) in the proof of Proposition 1 provides an explicit expression for a consistent variance estimator.

The fraction on the right-hand side of the bias expression in equation (9) can be estimated from observables using the calculated errors from equations (3), (5), and (8) and the fact that under the null

$$\text{cov}(\varepsilon_{t+1}^{PD}, \varepsilon_t^N) = \text{cov}(\varepsilon_{t+1}^{PD}, \mathbf{u}_t^N) - \text{cov}(\varepsilon_{t+1}^{PD}, u_t^N + \mu_t^N).$$

The bias $E(\hat{\rho}_T - \rho)$ in equation (9) is approximately given by $E(\hat{\rho}_T - \rho) \simeq -\frac{1+3\rho}{T}$ (see Marriott and Pope 1954). Since the true bias is a nonlinear function of ρ and the analytical linear approximation less precise in the relevant range of ρ close to 1,²⁴ we compute the bias $E(\hat{\rho}_T - \rho)$ using Monte Carlo integration.²⁵

Interestingly, our RE test $\mathbf{c}^N = c^N$ is less prone to small sample bias than tests for the significance of the individual regression coefficients ($\mathbf{c}^N = 0$ and $c^N = 0$). This follows from the proof of Proposition 1, which shows that

$$E(\hat{\mathbf{c}}_T^N - \mathbf{c}^N) = E(\hat{c}_T^N - c^N) + \frac{\text{cov}(\varepsilon_{t+1}^{PD}, \varepsilon_t^N)}{\text{var}(\varepsilon_t^{PD})} E(\hat{\rho}_T - \rho).$$

The previous equation implies that the small sample bias present in the individual estimate of c^N , i.e., $E(\hat{c}_T^N - c^N)$ cancels in the numerator of test statistic (7) when testing for $\mathbf{c}^N = c^N$.

The test outcomes associated with Proposition 1 are reported in Table 1A.²⁶ The table reports the bias-corrected point estimates of c^N and \mathbf{c}^N , as well as the bias-corrected p -values for the test based on Proposition 1.²⁷ We use the UBS, CFO, and Shiller surveys and consider various ways for extracting expectations from these surveys.²⁸ The point estimates always satisfy $\hat{c}^N > 0$ and $\hat{\mathbf{c}}^N < 0$. The difference between the two estimates is statistically significant at the 1 percent level in all cases, except when using the survey median from the CFO survey, where p -values are around 3 to 5 percent.²⁹

Overall, the test results in Table 1A provide strong evidence against the notion that survey expectations are compatible with rational expectations. Table 1A also shows that agents are overly optimistic when the PD ratio is high and overly pessimistic when the PD ratio is low. This suggests that current prices have an *excessive role* in influencing current return expectations. Clearly, if the asset price and survey data were generated by a rational expectations model—say, the models of Campbell and Cochrane (1999) or Bansal and Yaron (2004)—the tests in Table 1A would have been accepted.

²⁴ See, for instance, MacKinnon and Smith (1998, Figure 1).

²⁵ Given the estimated values of \overline{PD} , ρ , $\sigma_{\varepsilon^{PD}}^2$ we simulate 10,000 realizations of PD of length T , compute $\hat{\rho}$ for each realization, average over realizations to obtain an approximation for $E(\hat{\rho}_T)$, and compute the bias correction accordingly.

²⁶ We used 4 lags for Newey West estimator and we checked that results are robust to increasing the lag length up to 12 lags. For each considered survey, we use data on actual returns (or excess returns, or price growth) for the same time period for which survey data are available when computing the p -values.

²⁷ The p -values are computed using the bias corrected test statistic $\sqrt{T} \frac{\hat{\mathbf{c}}_T^N - \hat{c}_T^N - E(\hat{\mathbf{c}}_T^N - \hat{c}_T^N)}{\hat{\sigma}_{\mathbf{c}-c}}$.

²⁸ See online Appendix A1 for information on the data sources.

²⁹ We conjecture that the CFO provides less significant results because the sample starts in 2000:III, and thus does not include the upswing of the tech boom period, unlike the UBS sample. As a result, the CFO sample period displays less mean reversion in prices, which accounts for the fact that the estimates of \mathbf{c} are less negative and less significant.

TABLE 1A—RATIONAL EXPECTATIONS TEST, $H_0: c = \mathbf{c}$
(Stock Returns, Small Sample Bias Correction from Proposition 1)

| | Survey average | | | | Survey median | | | |
|--|----------------------|-------------------------------|---|-------------------------------------|----------------------|-------------------------------|---|-------------------------------------|
| | $\hat{c} \cdot 10^3$ | $\hat{\mathbf{c}} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{\mathbf{c}} - \hat{c})$ | p -value $H_0: c = \mathbf{c}$ | $\hat{c} \cdot 10^3$ | $\hat{\mathbf{c}} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{\mathbf{c}} - \hat{c})$ | p -value $H_0: c = \mathbf{c}$ |
| <i>Panel A. S&P 500, real returns</i> | | | | | | | | |
| UBS, > 100k, 1 yr, SPF | 0.58 | -2.46 | 0.432 | 0.0000 | 0.48 | -2.49 | 0.415 | 0.0000 |
| UBS, > 100k, 1 yr, Michigan | 0.57 | -2.46 | 0.452 | 0.0000 | 0.47 | -2.49 | 0.413 | 0.0000 |
| UBS, all, 1 yr, SPF | 0.57 | -2.46 | 0.424 | 0.0000 | 0.49 | -2.49 | 0.401 | 0.0000 |
| UBS, all, 1 yr, Michigan | 0.56 | -2.46 | 0.442 | 0.0000 | 0.48 | -2.49 | 0.433 | 0.0000 |
| CFO, 1 yr, SPF | 0.20 | -1.67 | 0.222 | 0.0011 | 0.25 | -1.37 | 0.325 | 0.0471 |
| CFO, 1 yr, Michigan | 0.27 | -1.67 | 0.200 | 0.0006 | 0.34 | -1.37 | 0.313 | 0.0362 |
| <i>Panel B. Dow Jones, real price growth</i> | | | | | | | | |
| Shiller, 1 yr, SPF | 0.26 | -1.22 | 0.235 | 0.0011 | 0.24 | -1.20 | 0.265 | 0.0015 |
| Shiller, 1 yr, Michigan | 0.33 | -1.22 | 0.232 | 0.0006 | 0.31 | -1.20 | 0.238 | 0.0007 |
| Shiller, 10 yrs, SPF | 4.73 | -7.25 | -1.367 | 0.0000 | 6.15 | -7.24 | -1.440 | 0.0000 |
| Shiller, 10 yrs, Michigan | 4.24 | -7.25 | -1.423 | 0.0000 | 5.65 | -7.24 | -1.462 | 0.0000 |

Notes: The table reports p -values for the rational expectations test from Section IIB for different survey sources and different ways to extract expectations from the surveys. Estimates and p -values are bias corrected as described in Section IIB and Proposition 1. \hat{c} is the estimate of c^N in equation (3) and $\hat{\mathbf{c}}$ the estimate of \mathbf{c}^N in equation (5). The column labeled Bias reports the small sample bias of $\hat{\mathbf{c}} - \hat{c}$ as implied by Proposition 1. The UBS and CFO surveys report return expectations for the S&P 500, the Shiller surveys report capital gain expectations for the Dow Jones Index. In the regressions, $R_{t,t+N}$ denotes returns, except for the Shiller survey, where it denotes capital gains. The columns labeled survey average compute expectations using the cross-sectional average of return survey expectations; the columns labeled survey median use the median survey expectation. In the first column, SPF and Michigan refer to different approaches to compute real expected returns, with the former using inflation expectations from the Survey of Professional Forecasters (SPF) and the latter using the Michigan survey; 1 yr and 10 yr refer to forecast horizons of 1 and 10 years, respectively; UBS, >100k indicates a restricted sample using only UBS survey participants with more than US\$100,000 in financial wealth; UBS, all indicates the use of all survey participants.

C. Additional Small Sample Adjustments

The closed-form expressions for the small sample bias derived in the previous section are useful for understanding the nature of the bias. At the same time, they fall short of completely addressing small sample issues. In particular, the result stated in Proposition 1(ii) relies on assuming that the regression residuals $u_t^N + \mu_t^N$ and \mathbf{u}_t^N in equations (3) and (5) are i.i.d. Yet, this is unlikely to hold in our application.

Consider first the residual \mathbf{u}_t^N . Under the null hypothesis of RE, we have $\mathbf{u}_t^N = u_t^N + \varepsilon_t^N$, where $\varepsilon_t^N \equiv R_{t,t+N} - E_t R_{t,t+N}$ is a prediction error from the true data-generating process. For short prediction horizons ($N = 1$), ε_t^N is indeed serially uncorrelated, but for longer horizons ($N > 1$), the residuals ε_t^N denote forecast errors for overlapping prediction horizons. Since we use quarterly data and prediction horizons between one and ten years, serial correlation of \mathbf{u}_t^N is a relevant concern.

Next, consider the residual u_t^N . Under the null hypothesis, it captures information that agents use to predict future returns over and above the PD ratio. The variables capturing such information can be expected to be themselves serially correlated.³⁰

³⁰For example, in a model with capital accumulation, u_t^N is a function of the capital stock, which is highly serially correlated.

Indeed, once we take into account serial correlation below, we find a quarterly persistence for u_t^N between 0.77 and 0.93.

An additional concern with the small sample result in Proposition 1 is that the variance $\hat{\sigma}_{c-c}$ is usually underestimated in regressions involving highly serially correlated variables. Ferson, Sarkissian, and Simin (2003) and Campbell and Yogo (2006) show that this leads to spurious regression problems that cause the null hypothesis to be rejected too often in the kind of regressions we are dealing with.

To address these issues, this section constructs p -values using a Monte Carlo procedure to find the actual distribution of the statistic $\sqrt{T}(\hat{\mathbf{c}}^N - \hat{\mathbf{c}}^N)/\hat{\sigma}_{c-c}$. The aim is to build a model for returns that is as close as possible to the one used in Stambaugh (1999), but that allows for serial correlation of the error terms. To this end, we consider in addition to equation (8) an equation for returns of the form

$$(10) \quad R_{t,t+N} = \mathbf{A}^N + \mathbf{C}^N \frac{P_t}{D_t} + \mathbf{U}_{t+N}^N,$$

with given constants \mathbf{A}^N and \mathbf{C}^N . We allow \mathbf{U}_t^N to be serially correlated by specifying it as an AR(1) process. Since the innovations ε_t^{PD} to the PD ratio in equation (8) are likely also a key component of the innovation to returns, we allow the innovations to \mathbf{U}_t^N to be correlated with ε_t^{PD} and consider

$$(11) \quad \mathbf{U}_t^N = \chi \mathbf{U}_{t-1}^N + \eta_t + \lambda \varepsilon_t^{PD}$$

for $\eta_t \sim N(0, \sigma_\eta^2)$, independent of ε_j^{PD} at all dates j , and for given constants χ and λ satisfying $|\chi| < 1$.

Notice that equation (10) is not a special case of equation (5) in Section IIB. The reason is that \mathbf{U}_{t+N}^N is correlated with ε_{t+N-j}^{PD} for $j \geq 0$ and thus correlated with PD_t :

$$E(\mathbf{U}_{t+N}^N PD_t) = \frac{(\chi)^N \lambda \sigma_{\varepsilon^{PD}}^2}{1 - (\chi\rho)^2}.$$

As a result, the regression coefficients $(\mathbf{a}^N, \mathbf{c}^N)$ in equation (5) do not satisfy $\mathbf{a}^N = \mathbf{A}^N$, $\mathbf{c}^N = \mathbf{C}^N$, whenever $\chi\lambda \neq 0$. It would thus be incorrect to plug our estimates of \mathbf{a}^N , \mathbf{c}^N into equation (10) for the purpose of running the Monte Carlo simulations.

To estimate the parameters in equations (10) and (11), we proceed as follows. We lag equation (10) by one period, multiply by χ and subtract it from equation (10). This delivers

$$(12) \quad R_{t,t+N} = \mathbf{A}^N(1 - \chi) + \chi R_{t-1,t+N-1} + \mathbf{C}^N \frac{P_t}{D_t} - \chi \mathbf{C}^N \frac{P_{t-1}}{D_{t-1}} + \lambda \varepsilon_{t+N}^{PD} + \eta_{t+N},$$

which can be estimated using nonlinear least squares and the observed explanatory variables $\left(R_{t-1,t+N-1}, \frac{P_t}{D_t}, \frac{P_{t-1}}{D_{t-1}}, \hat{\varepsilon}_{t+N}^{PD}\right)$, because these explanatory variables are orthogonal to η_{t+N} . We thus have consistent and efficient estimates for χ , λ , σ_η^2 , \mathbf{A}^N , and \mathbf{C}^N . We plug these estimates into equations (10) and (11) to simulate $R_{t,t+N}$.

TABLE 1B—RATIONAL EXPECTATIONS TEST, $H_0: c \leq \mathbf{c}$
 (Stock Returns, One-Sided Test, Small Sample Corrections with Serially Correlated Errors
 and Nonlinear Least Squares Estimates from Section IIC)

| | χ | Survey average | | | | Survey median | | | |
|--|--------|----------------------|----------------------|--|-------------------------------|----------------------|----------------------|--|-------------------------------|
| | | $\hat{c} \cdot 10^3$ | $\hat{c} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{c} - \hat{c})$ | p -value $H_0: c \leq c$ | $\hat{c} \cdot 10^3$ | $\hat{c} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{c} - \hat{c})$ | p -value $H_0: c \leq c$ |
| <i>Panel A. S&P 500, real returns</i> | | | | | | | | | |
| UBS*, >100k, 1 yr, SPF | 0.82 | 2.80 | -0.64 | 0.054 | 0.014 | 2.68 | -0.62 | 0.062 | 0.013 |
| UBS*, >100k, 1 yr, Michigan | 0.82 | 2.79 | -0.64 | 0.054 | 0.016 | 2.67 | -0.62 | 0.071 | 0.014 |
| UBS*, all, 1 yr, SPF | 0.82 | 2.78 | -0.64 | 0.072 | 0.013 | 2.69 | -0.62 | 0.063 | 0.013 |
| UBS*, all, 1 yr, Michigan | 0.82 | 2.77 | -0.64 | 0.072 | 0.015 | 2.68 | -0.62 | 0.055 | 0.014 |
| CFO, 1 yr, SPF | 0.94 | 5.83 | 3.19 | -0.568 | 0.046 | 6.59 | 4.07 | -0.550 | 0.111 |
| CFO, 1 yr, Michigan | 0.94 | 5.90 | 3.19 | -0.568 | 0.044 | 6.67 | 4.07 | -0.520 | 0.107 |
| <i>Panel B. Dow Jones, real price growth</i> | | | | | | | | | |
| Shiller, 1 yr, SPF | 0.92 | 4.68 | 2.34 | -0.621 | 0.020 | 4.68 | 2.38 | -0.627 | 0.018 |
| Shiller, 1 yr, Michigan | 0.92 | 4.74 | 2.34 | -0.624 | 0.016 | 4.74 | 2.38 | -0.627 | 0.017 |
| Shiller, 10 yrs, SPF | 0.76 | 8.07 | -2.65 | -0.130 | 0.016 | 9.45 | -2.66 | -0.200 | 0.014 |
| Shiller, 10 yrs, Michigan | 0.76 | 7.48 | -2.65 | 0.200 | 0.023 | 8.85 | -2.66 | -0.200 | 0.019 |

Notes: The column labeled χ reports the estimated persistence parameter of the return residuals \mathbf{U}_t^N in equation (11). See the notes to Table 1A for further information.

To compute expected returns under the null hypothesis of RE, we compute the true expectation of returns, which are given by

$$E_t(R_{t,t+N}) = \mathbf{A}^N + \mathbf{C}^N \frac{P_t}{D_t} + (\chi)^N \mathbf{U}_t^N.$$

Using these results, we can simulate all the variables involved in equations (3) and (5), compute the statistic $\sqrt{T}(\hat{\mathbf{c}}^N - \hat{\mathbf{c}}^N) / \hat{\sigma}_{c-c}$ for each simulation and study its small sample distribution, using the sample sizes of the considered survey source. We then compute the probability that $\sqrt{T}(\hat{\mathbf{c}}^N - \hat{\mathbf{c}}^N) / \hat{\sigma}_{c-c}$ in the Monte Carlo simulations is smaller than the corresponding value we find for the data. This provides a p -value for the one-sided test of RE, when the alternative hypothesis is $c^N > \mathbf{c}^N$, i.e., that survey returns respond more strongly to the PD ratio than actual returns.

Table 1B reports the outcomes of this procedure. The second column in the table reports the estimated value for χ . It shows that the residuals \mathbf{U}_t^N in equation (11) are indeed serially correlated. We find that this leads to considerable spurious regression problems, as the standard deviation of the test statistic $\sqrt{T}(\hat{\mathbf{c}}^N - \hat{\mathbf{c}}^N) / \hat{\sigma}_{c-c}$ is indeed around 2 to 3 times larger than its asymptotic value of 1.

Table 1B also reports the bias corrected estimates $\hat{\mathbf{c}}^N$ and $\hat{\mathbf{c}}^N$.³¹ Compared to the results in Table 1A, the estimates for $\hat{\mathbf{c}}^N$ are considerably less negative; the ones for the CFO and Shiller 1 year sample even become positive. Yet, the bias-corrected estimates for $\hat{\mathbf{c}}^N$ in Table 1B also become more positive when compared to the ones reported in Table 1A. Therefore, despite the spurious regression problems, which cause an increase in the true variance of the test statistic, the RE hypothesis is

³¹The point estimates correct for small sample bias using the mean of the estimator found in the Monte Carlo simulations with serially correlated errors and using the fact that $\mathbf{c}^N = \mathbf{C}^N + (\chi)^N \lambda(1 - \rho^2) / (1 - (\chi\rho)^2)$.

TABLE 1C—RATIONAL EXPECTATIONS TEST, $H_0: c \leq c$
*(Excess Stock Returns, One-Sided Test, Small Sample Corrections with Serially Correlated Errors
and Nonlinear Least Squares Estimates from Section IIC)*

| | χ | Survey average | | | | Survey median | | | |
|--|--------|----------------------|----------------------|--|-------------------------------|----------------------|----------------------|--|-------------------------------|
| | | $\hat{c} \cdot 10^3$ | $\hat{c} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{c} - \hat{c})$ | p -value $H_0: c \leq c$ | $\hat{c} \cdot 10^3$ | $\hat{c} \cdot 10^3$ | bias $\cdot 10^3$ $-E(\hat{c} - \hat{c})$ | p -value $H_0: c \leq c$ |
| <i>Panel A. S&P 500, real returns</i> | | | | | | | | | |
| UBS, >100k, 1 yr, SPF | 0.77 | 2.00 | -1.11 | 0.153 | 0.051 | 1.89 | -1.10 | 0.154 | 0.058 |
| UBS, >100k, 1 yr, Michigan | 0.77 | 2.00 | -1.11 | 0.157 | 0.056 | 1.89 | -1.10 | 0.158 | 0.054 |
| UBS, all, 1 yr, SPF | 0.77 | 1.98 | -1.11 | 0.161 | 0.056 | 1.90 | -1.10 | 0.154 | 0.057 |
| UBS, all, 1 yr, Michigan | 0.77 | 1.98 | -1.11 | 0.154 | 0.054 | 1.90 | -1.10 | 0.160 | 0.062 |
| CFO, 1 yr, SPF | 0.93 | 5.58 | 3.00 | -0.532 | 0.067 | 6.44 | 4.12 | -0.517 | 0.155 |
| CFO, 1 yr, Michigan | 0.93 | 5.59 | 3.00 | -0.540 | 0.065 | 6.45 | 4.12 | -0.522 | 0.151 |
| <i>Panel B. Dow Jones, real price growth</i> | | | | | | | | | |
| Shiller, 1 yr, SPF | 0.91 | 4.01 | 1.85 | -0.562 | 0.028 | 4.01 | 1.84 | -0.548 | 0.028 |
| Shiller, 1 yr, Michigan | 0.91 | 4.00 | 1.85 | -0.547 | 0.030 | 4.01 | 1.84 | -0.540 | 0.032 |
| Shiller, 10 yrs, SPF | 0.78 | 6.30 | -5.16 | 0.151 | 0.030 | 7.68 | -5.15 | 0.175 | 0.023 |
| Shiller, 10 yrs, Michigan | 0.78 | 6.14 | -5.16 | 0.383 | 0.033 | 7.52 | -5.15 | -0.214 | 0.026 |

Notes: The column labeled χ reports the estimated persistence parameter of the return residuals U_t^N in equation (11). See the notes to Table 1A for further information.

soundly rejected.³² The level of the rejection is now considerably lower than the one reported in Table 1A, but still highly significant. Since the involved sample lengths are not very large, this is a remarkable result.

Table 1C repeats the analysis when letting $R_{t,t+N}$ denote excess stock returns rather than stock returns.³³ We construct excess return expectations following Bacchetta, Mertens, and Van Wincoop (2009), i.e., we assume that the N period ahead risk-free interest rate is part of agents' information set and subtract it from the (expected) stock return.³⁴ We find that the strength of the rejection of RE is then somewhat lower when compared to Table 1B. Still, for most survey sources one obtains p -values near or below 5 percent. The somewhat lower p -values show that the approach based on plain stock returns, as reported in Table 1B, offers a slightly more powerful test of the RE hypothesis. Interestingly, this is in line with the main hypothesis of this paper, namely, that agents form their expectations about stock prices by extrapolating the behavior of past price growth. Under this hypothesis, subtracting the risk-free interest rate adds noise to the independent variables on the LHS of regression equations (3) and (5), which is consistent with the observed decrease in the significance levels.

D. How Models of Learning May Help

This section illustrates that a simple *adaptive* approach to forecasting stock prices is a promising alternative to explain the joint behavior of survey expectations and stock price data.

³² As in Table 1A, the rejection is less strong for the CFO survey, especially when using the survey median. See footnote 29 for a discussion for the potential reasons behind this result.

³³ For the Shiller survey, which reports price growth expectations, $R_{t,t+N}$ now denotes excess prices growth.

³⁴ Following Bacchetta, Mertens, and Van Wincoop (2009), we use the constant maturity interest rates available from the FRED database at the St. Louis Federal Reserve Bank.

Figure 2 shows that the peaks and troughs of the PD ratio are located very closely to the peaks and troughs of investors' return expectations. This suggests that agents become optimistic about future capital gains whenever they have observed high capital gains in the past. Such behavior can be captured by models where agents' expectations are influenced by past experience, prompting us to temporarily explore the assumption that the log of agents' subjective conditional capital gain expectations $\ln \tilde{E}_t[P_{t+1}/P_t]$ evolves according to the following adaptive prediction model:

$$(13) \quad \ln \tilde{E}_t[P_{t+1}/P_t] = (1 - g) \ln \tilde{E}_{t-1}[P_t/P_{t-1}] + g \ln P_{t-1}/P_{t-2},$$

where $g > 0$ indicates how strongly capital gain expectations are updated in the direction of past price growth observations. While equation (13) may appear ad hoc, we show in Section V how a very similar equation can be derived from Bayesian belief updating in a setting where agents estimate the persistent component of price growth from the data. Note that equation (13) incorporates price growth observations only with a lag, in line with the theoretical model that we consider later on.

One can feed into equation (13) the historical price growth data of the S&P 500 over the postwar period. Together with an assumption about capital gain expectations at the start of the sample, this delivers a time series of implied capital gain expectations $\ln \tilde{E}_t[P_{t+1}/P_t]$ that can be compared to the expectations from the UBS survey.³⁵ Setting the unobserved initial price growth expectations in 1946:I equal to 0 percent, Figure 3 reports the outcome of this procedure, when estimating the gain parameter g using nonlinear least squares to minimize the distance between the expectations implied by equation (13) and the observed survey expectations.³⁶ The resulting point estimate is given by $g = 0.0264$ and has a standard error equal to 0.00168. Figure 3 shows that the adaptive prediction model captures the behavior of UBS expectations extremely well: the correlation between the two series is equal to +0.91.

There also exists a strong positive relationship between the PD ratio and the capital gains expectations implied by equation (13). Figure 4 documents this relationship for the entire postwar period by plotting the joint distribution of the capital gains expectations (as implied by equation (13)) and the PD ratio in the data.³⁷ When regressing the PD ratio on a constant and the expectations of the adaptive prediction model, one obtains an R^2 coefficient of 0.43; using also the square of the expectations, the R^2 rises further to 0.48.³⁸

Interestingly, the relationship between implied price growth expectations and the PD ratio depicted in Figure 4 seems to have shifted upward after the year 2000, as indicated by the squared icons in the figure. Indeed running the previous regressions on the linear and squared expectations separately before and after the year 2000, one

³⁵ We transform the UBS survey measures of return expectations into a measure of price growth expectations using the identity $R_{t+1} = \frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} = \frac{P_{t+1}}{P_t} + \beta^D \frac{D_t}{P_t}$ where β^D denotes the expected quarterly growth rate of dividends that we set equal to the sample average of dividend growth over 1946:I–2012:I, i.e., $\beta^D = 1.0048$. Results regarding implied price growth are very robust toward changing β^D to alternative empirically plausible values.

³⁶ The figure reports log quarterly expected growth rates for real stock prices.

³⁷ As before, we set the unobserved initial price growth expectations in 1946:I equal to 0 percent.

³⁸ The p -values for the coefficients on the linear and squared expectations are all statistically significant at the 1 percent level.

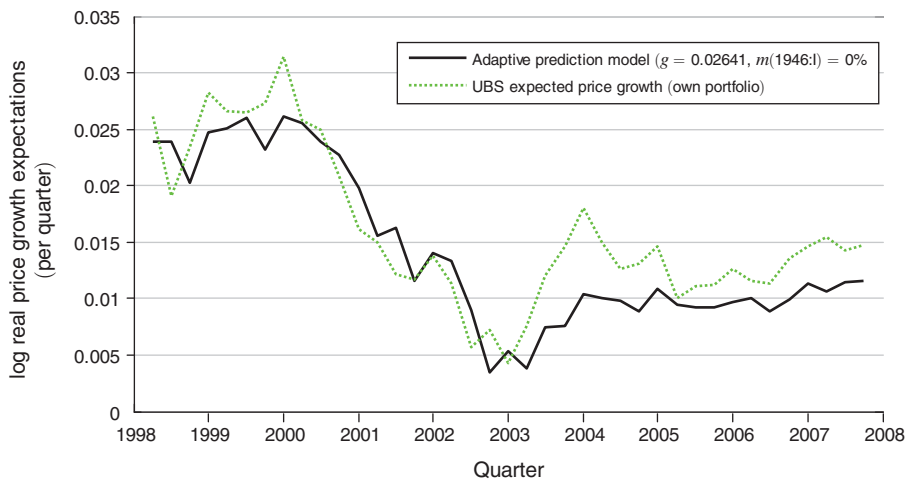


FIGURE 3. REAL PRICE GROWTH EXPECTATIONS IMPLIED BY THE UBS SURVEY EXPECTATIONS AND BY THE ADAPTIVE PREDICTION MODEL (Equation (13)), 1998:II–2007:IV

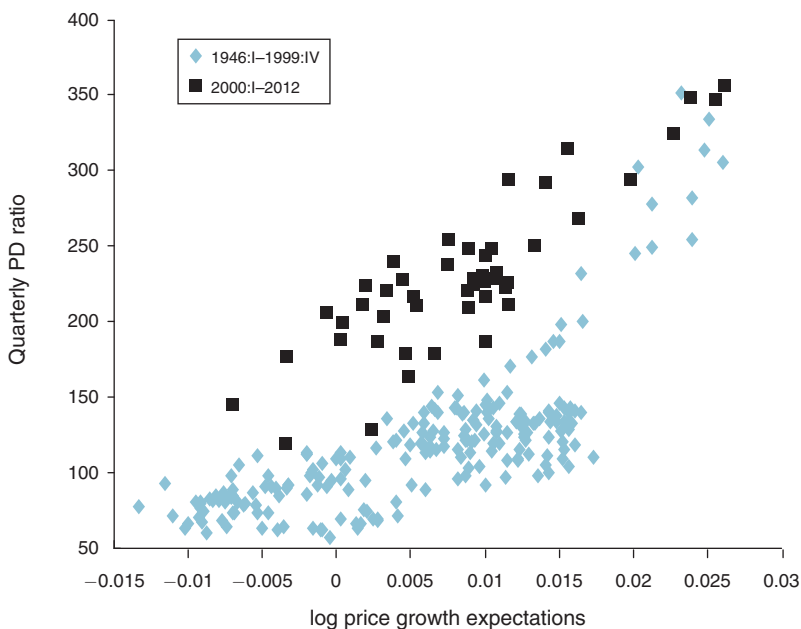


FIGURE 4. QUARTERLY PD RATIO OF THE S&P 500 GRAPHED AGAINST THE QUARTERLY REAL PRICE GROWTH EXPECTATIONS IMPLIED BY THE ADAPTIVE PREDICTION MODEL (Equation (13)), FOR THE SAME QUARTER, 1946:I–2012:I

obtains much higher R^2 values, namely 0.72 and 0.77, respectively. We will come back to this issue in Section IX.

Overall, it emerges from Figure 4 that variations in expected capital gains $\tilde{E}_t[P_{t+1}/P_t]$ can account—in a purely statistical sense—for a large share of the variability in the postwar PD ratio. This suggests that an asset-pricing model consistent with equation (13), which additionally predicts a positive relationship between the

PD ratio and subjective expectations about future capital gains, has a good chance of replicating the observed positive comovement between price growth expectations and the PD ratio. The next sections spell out the microfoundations of such a model.

III. A Simple Asset-Pricing Model

Consider an endowment economy populated by a unit mass of infinitely lived agents $i \in [0, 1]$ with time-separable preferences. Agents trade one unit of a stock in a competitive stock market. They earn each period an exogenous nondividend income $W_t \geq 0$ that we refer to as *wages* for simplicity. Stocks deliver the exogenous dividend $D_t \geq 0$. Dividend and wage incomes take the form of perishable consumption goods.

The Investment Problem.—Investor i solves

$$(14a) \quad \max_{\{C_t^i \geq 0, S_t^i\}_{t=0}^{\infty}} E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t u(C_t^i)$$

$$\text{subject to} \quad S_t^i P_t + C_t^i = S_{t-1}^i (P_t + D_t) + W_t \quad \text{for all } t \geq 0,$$

$$(14b) \quad \underline{S} \leq S_t^i \leq \bar{S},$$

$$S_{-1}^i = 1,$$

where C^i denotes consumption, u the instantaneous utility of the consumer, assumed to be continuous, differentiable, increasing and strictly concave, S^i the agent's stockholdings, which are subject to upper and lower limits such that $-\infty < \underline{S} < 1 < \bar{S} < \infty$, and $P \geq 0$ the (ex-dividend) price of the stock; \mathcal{P}^i denotes the agent's subjective probability measure, which may or may not satisfy the rational expectations hypothesis. Further details of \mathcal{P}^i will be specified below.

To simplify the exposition, we do not explicitly consider trade in risk-free bonds (B_t^i) in problem (14), instead impose from the outset that actual bond prices and agents' subjective beliefs about future bond prices are such that agents hold and plan to hold zero bonds at all times ($B_t^i \equiv 0$). When bonds are in zero net supply, these beliefs are in line with the equilibrium outcomes, when all agents are identical or when agents face the borrowing constraint $B_t^i \geq 0$.³⁹

Dividend and Wage Income.—As is standard in the literature, we assume that dividends grow at a constant rate and that dividend growth innovations are unpredictable:

$$(15) \quad \ln D_t = \ln \beta^D + \ln D_{t-1} + \ln \varepsilon_t^D,$$

where $\beta^D \geq 1$ denotes gross mean dividend growth and $\ln \varepsilon_t^D$ an i.i.d. growth innovation described further below.

³⁹In the latter case, the *most patient agent* prices the bond, i.e., letting R^b denote the gross real return on a one-period risk-free bond, we have $1/R^b = \sup_{i \in [0, 1]} \delta E_t^{\mathcal{P}^i} u'(C_{t+1}^i) / u'(C_t^i)$.

We also specify an exogenous wage income process W_t , which is chosen such that the resulting aggregate consumption process $C_t = W_t + D_t$ is empirically plausible. First, in line with Campbell and Cochrane (1999), we set the standard deviation of consumption growth to be 1/7 of the standard deviation of dividend growth. Second, again following these authors, we set the correlation between consumption and dividend growth equal to 0.2. Third, we choose a wage process in the model such that the average consumption-dividend ratio in the model ($E[C_t/D_t]$) equals the average ratio of personal consumption expenditure to net dividend income in US postwar data and displays persistence similar to that observed in the data. All this can be parsimoniously achieved using the following wage income process:

$$(16) \quad \ln\left(1 + \frac{W_t}{D_t}\right) = (1 - p) \ln(1 + \rho) + p \ln\left(1 + \frac{W_{t-1}}{D_{t-1}}\right) + \ln \varepsilon_t^W,$$

where $1 + \rho$ is the average consumption-dividend ratio and $p \in [0, 1)$ its quarterly persistence. The innovations are given by

$$(17) \quad \begin{pmatrix} \ln \varepsilon_t^D \\ \ln \varepsilon_t^W \end{pmatrix} \sim iin\left(-\frac{1}{2} \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}\right),$$

with $E\varepsilon_t^D = E\varepsilon_t^W = 1$. Given the variance of dividend growth σ_D^2 , which can be estimated from dividend data, one can use σ_{DW} and σ_W^2 to impose the desired volatility of consumption growth and the desired correlation with dividend growth. Choosing $\rho = .22$, one obtains the observed average consumption-dividend ratio in the data and setting p to a value close to 1, one replicates the observed persistence of the consumption-dividend ratio in the data. Online Appendix A4 provides further details.

The Agents' Underlying Probability Space.—Agents hold a set of subjective probability beliefs about all payoff-relevant variables that are beyond their control. In addition to fundamental variables (dividends and wages), agents perceive competitive stock prices as beyond their control. Therefore, the belief system also specifies probabilities about prices. Formally, letting Ω denote the space of possible realizations for infinite sequences, a typical element $\omega \in \Omega$ is given by $\omega = \{P_t, D_t, W_t\}_{t=0}^\infty$. As usual, Ω^t then denotes the set of all possible (nonnegative) price, dividend, and wage histories from period zero up to period t and ω^t its typical element. The underlying probability space for agents' beliefs is then given by $(\Omega, \mathcal{B}, \mathcal{P}^i)$ with \mathcal{B} denoting the corresponding σ -Algebra of Borel subsets of Ω , and \mathcal{P}^i a probability measure over (Ω, \mathcal{B}) .

The agents' plans will be contingent on the history ω^t , i.e., the agent chooses state-contingent consumption and stockholding functions

$$(18) \quad C_t^i: \Omega^t \rightarrow \mathbb{R}_+,$$

$$(19) \quad S_t^i: \Omega^t \rightarrow [\underline{S}, \bar{S}].$$

The fact that C^i and S^i depend on price realizations is a consequence of optimal choice under uncertainty, given that agents consider prices to be exogenous random variables.

The previous setup is general enough to accommodate situations where agents learn about the stochastic processes governing the evolution of prices, dividends, and wages. For example, \mathcal{P}^i may arise from a stochastic process describing the evolution of these variables that contains unknown parameters or hidden variables about which agents hold prior beliefs. The presence of unknown parameters or hidden variables then implies that agents update their beliefs using the observed realizations of prices, dividends, and wages. A particular example of this kind will be presented in Section V when we discuss learning about stock price behavior.

The probability space defined above is more general than that specified in a RE analysis of the model, where Ω contains usually only the variables that are exogenous to the model (in this case D_t and W_t), but not variables that are endogenous to the model and exogenous to the agent only (in this case P_t). In the RE literature, the Bayesian RE literature, or the literature modeling robustness concerns, agents are assumed to know the equilibrium pricing function $P_t((D, W)^t)$ mapping histories of dividends and wages into a market price. Prices then carry only redundant information and can be excluded from the probability space without loss of generality. The more general formulation we entertain here allows us to consider agents who do not know exactly which price materializes given a particular history of dividends and wages; our agents do have a view about the distribution of P_t conditional on $(D, W)^t$, but in their minds this is a proper distribution, not a point mass as in the RE case. Much akin to academic economists, investors in our model have not converged on a single asset-pricing model that associates one market price with a given history of exogenous fundamentals.

Parametric Utility Function.—To obtain closed-form solutions, we consider in the remaining part of the paper the utility function

$$(20) \quad u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \quad \text{with } \gamma > 1.$$

We furthermore assume that

$$(21) \quad \delta \beta^{RE} < 1,$$

where $\beta^{RE} = (\beta^D)^{1-\gamma} E[(\varepsilon_t^W)^{-\gamma} (\varepsilon_t^D)^{1-\gamma}]$. This insures existence of an equilibrium under rational price expectations.

Existence of a Recursive Solution.—Since solving the optimization problem (14) for general (potentially nonrational) price beliefs is nonstandard, it is worth pointing out that this problem is well defined. Existence of a maximum is guaranteed by the stock limits (14b), which ensure that the choice of stocks is made over a compact set, in combination with a bounded continuous objective function (20), satisfying $u(C_t) \leq 0$. Sufficiency of first-order conditions (FOC) is guaranteed because the agents' problem is concave. Given the subjective price beliefs introduced in the

remaining part of the paper, agents' posterior will follow a recursive form. Using a standard normalization of the utility function and of the budget constraint, one can then guarantee that the optimal solution to (14) takes the form

$$(22) \quad S_t^i = S^i\left(S_{t-1}^i, \frac{P_t}{D_t}, \frac{W_t}{D_t}, m_t^i\right),$$

where m_t^i is a sufficient statistic characterizing the subjective distributions about future values of $\left(\frac{D_{t+j}}{D_{t+j-1}}, \frac{P_{t+j}}{D_{t+j}}, \frac{W_{t+j}}{D_{t+j}}\right)$ for $j > 0$.⁴⁰ See online Appendix A5 for details.

IV. Rational Expectations Equilibrium

As a point of reference, we determine the stock price under the assumption of RE. For the limiting cases with an i.i.d. consumption-dividend ratio ($p = 0$) or with a unit root in this ratio ($p \rightarrow 1$), one can derive simple closed-form expressions for the RE stock price: see online Appendix A6.

PROPOSITION 2: *If agents hold rational expectations and if price expectations satisfy the usual transversality condition (stated explicitly in online Appendix A6), then the RE equilibrium price dividend ratio is*

$$(23) \quad \frac{P_t^{RE}}{D_t} = \frac{\delta \beta^{RE}}{1 - \delta \beta^{RE}} \quad \text{for } p \rightarrow 1,$$

$$(24) \quad \frac{P_t^{RE}}{D_t} = b \frac{\delta \tilde{\beta}^{RE}}{1 - \delta \tilde{\beta}^{RE}} (\varepsilon_t^W)^\gamma \quad \text{for } p = 0,$$

where $\beta^{RE} \equiv (\beta^D)^{1-\gamma} E[(\varepsilon_t^W)^{-\gamma} (\varepsilon_t^D)^{1-\gamma}]$, $b \equiv E[(\varepsilon_t^W)^{-\gamma} (\varepsilon_t^D)^{1-\gamma}] e^{\gamma(1-\gamma)\frac{\sigma_b^2}{2}}$, and $\tilde{\beta}^{RE} \equiv (\beta^D)^{1-\gamma} e^{\gamma(\gamma-1)\frac{\sigma_b^2}{2}}$.

The previous result shows that under RE the equilibrium PD ratio inherits the persistence properties of the consumption-dividend ratio process. Specifically, for $p = 0$, the PD ratio is an i.i.d. process, thus, fails to match the persistence of the PD ratio observed in the data. For the empirically more plausible case with a persistent consumption-dividend ratio (p close to 1), the PD ratio also becomes persistent but volatility then strongly falls. In the limit $p \rightarrow 1$, which is the case considered in Campbell and Cochrane (1999), the RE PD ratio becomes a constant. Price growth is then given by

$$(25) \quad \ln P_{t+1}^{RE} - \ln P_t^{RE} = \ln \beta^D + \ln \varepsilon_{t+1}^D,$$

⁴⁰The solution for optimal consumption plans follows from (22) and the flow budget constraint.

so that one-step-ahead price growth and return expectations are constant over time. Moreover, as long as p is bounded below 1, the RE equilibrium implies a *negative* correlation between the PD ratio and expected future returns, contrary to what is evidenced by survey data.⁴¹ The RE equilibrium thus has difficulties in jointly matching the persistence and volatility of the PD ratio and the behavior of survey returns.

V. Learning about Capital Gains and Internal Rationality

This section specifies agents' beliefs. Once one departs from rational expectations, these beliefs become part of the microfoundations of the model. The benchmark beliefs presented in this section aim at specifying price beliefs in a parsimonious way such that they have a chance of replicating the behavior of survey expectations. We study alternative and more elaborate belief formulations in Sections XB and XC.

A. Agents' Belief System: Benchmark Specification

To focus on the effects of learning about price behavior, we assume that agents know the processes (15)–(17), i.e., agents hold rational dividend and wage expectations. This sets us apart from the Bayesian RE literature, which focuses on learning about the behavior of exogenous fundamentals.

We also have to specify price beliefs. Our specification is motivated by the following observations. Under RE, price growth is approximately equal to dividend growth, whenever the consumption-dividend ratio is persistent (p close to 1): see Proposition 2. Price growth in the data, however, can persistently outstrip dividend growth, thereby giving rise to a persistent increase in the PD ratio and an asset price boom (see Figure 1); conversely, it can fall persistently short of dividend growth and give rise to a price bust. This behavior of actual asset prices suggests that it is of interest to relax agents' beliefs about price growth behavior. Indeed, in view of the actual behavior of asset prices, agents may entertain a more general model of price behavior, incorporating the possibility that the growth rate of prices persistently exceeds/falls short of the growth rate of dividends. To the extent that the equilibrium asset prices implied by these beliefs display such data-like behavior, agents' beliefs will be generically validated.

In line with the previous discussion, our benchmark assumption is that agents perceive prices to evolve according to

$$(26) \quad \ln P_{t+1} - \ln P_t = \ln \beta_{t+1} + \ln \varepsilon_{t+1},$$

where ε_{t+1} denotes a transitory shock to price growth and β_{t+1} a persistent price growth component that drifts slowly over time according to

$$(27) \quad \ln \beta_{t+1} = \ln \beta_t + \ln \nu_{t+1}.$$

⁴¹ Online Appendix A6 derives the general expression for the RE price for the case $p \in (0, 1)$.

For simplicity, we assume that agents perceive the innovations $\ln \varepsilon_t$ and $\ln \nu_t$ to be jointly normally distributed according to

$$(28) \quad \begin{pmatrix} \ln \varepsilon_t \\ \ln \nu_t \end{pmatrix} \sim iin \left(\begin{pmatrix} -\frac{\sigma_\varepsilon^2}{2} \\ -\frac{\sigma_\nu^2}{2} \end{pmatrix}, \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\nu^2 \end{pmatrix} \right).$$

Since agents observe stock price growth, but do not separately observe the persistent and transitory subcomponents driving it, the previous setup defines a filtering problem in which agents need to decompose observed price growth into its persistent and transitory elements, so as to forecast optimally.⁴²

The presented belief setup has a number of appealing features. First, it allows capturing the fact that stock price growth displays large transitory fluctuations by imposing $\sigma_\nu^2 \ll \sigma_\varepsilon^2$. In the limiting case where σ_ν^2 is close to 0, the persistent price growth component behaves almost like a constant, as in the RE solution with $p \rightarrow 1$. The subjective price beliefs specified above can thus be interpreted as a small deviation from RE equilibrium beliefs. Second, the benchmark setup can capture periods with sustained increases in the *PD* ratio ($\beta_{t+1} > \beta^D$) and sustained decreases ($\beta_{t+1} < \beta^D$), in line with the behavior of the *PD* ratio in the data. To the extent that the *PD* ratio in the model also displays such behavior, agents' beliefs about the presence of a persistent price growth component will also be generically validated by the model outcome. Third, as we shall show in the next section, the belief specification implies that agents perceived price growth expectations will approximately evolve according to equation (13), which captures the time series behavior of survey expectations.

As with any assumption about model primitives, one can entertain other plausible alternatives. Section XB explores alternative ways to specify the belief system, which incorporate mean reversion in the *PD* ratio.

B. Efficiency of Stock Prices and Internal Rationality

Among academics there appears to exist a widespread belief that rational behavior and knowledge of the fundamental processes (dividends and wages in our case) jointly dictate a certain process for stock prices and thus the price beliefs that agents can rationally entertain.⁴³ This view stipulates that rational behavior implies knowledge that current stock prices must equal a discounted sum of dividends. Individual rationality and rational expectations about fundamentals would then provide investors with knowledge of the equilibrium pricing function (as is assumed under RE or Bayesian RE), so that postulating subjective price beliefs, e.g., those specified in the previous section, would be inconsistent with the assumption of optimal behavior on the part of agents.

⁴²Note that we do not incorporate mean-reversion into price growth beliefs in our benchmark setting. This is for simplicity, as we wish to consider the most parsimonious way to include perceived booms and busts. In addition, we seek to determine the model-endogenous forces that lead to a reversal of boom and bust dynamics, i.e., we do not want to obtain reversals because they are hard-wired into beliefs. We extend the setup to one with mean reversion in Section XB.

⁴³We often received this reaction during seminar presentations.

This view is correct only in special cases. Considering the case with risk-neutral agents, Adam and Marcet (2011) show that it is correct only if agents are homogeneous and do not face trading constraints. In a setting with heterogeneous risk-neutral agents and trading constraints, it fails to be correct. As we show below, agents in our model are *internally rational* even in the homogeneous agent case: their behavior is optimal given an internally consistent system of subjective beliefs about variables that are beyond their control, which includes prices. All that is needed is a concave utility function.⁴⁴

To illustrate this point, consider first risk-neutral agents with rational dividend expectations and ignore limits to stock holdings. Forward-iteration on the agents' optimality condition

$$(29) \quad u'(C_t^i) = \delta E_t^{\mathcal{P}^i} \left[u'(C_{t+1}^i) \frac{P_{t+1} + D_{t+1}}{P_t} \right]$$

and risk-neutrality deliver the present value relationship

$$P_t = E_t \left[\sum_{i=1}^T \delta^i D_{t+i} \right] + \delta^T E_t^{\mathcal{P}^i} [P_{t+T}],$$

which is independent of the agent's own choices. Provided agents' price beliefs satisfy a standard transversality condition ($\lim_{T \rightarrow \infty} \delta^T E_t^{\mathcal{P}^i} [P_{t+T}] = 0$ for all i), each rational agent would conclude that there must be a degenerate joint distribution for prices and dividends given by

$$(30) \quad P_t = E_t \left[\sum_{i=1}^{\infty} \delta^i D_{t+i} \right] \text{ a.s.}$$

Since the RHS of the previous equation is fully determined by dividend expectations, the beliefs about the dividend process deliver the price process compatible with optimal behavior. In such a setting, it would be plainly inconsistent with optimal behavior to assume the subjective price beliefs (26)–(27).⁴⁵

Next, consider a concave utility function $u(\cdot)$ satisfying standard Inada conditions. Forward iteration on (29) and assuming an appropriate transversality condition then delivers

$$(31) \quad P_t u'(C_t) = E_t^{\mathcal{P}} \left[\sum_{j=1}^{\infty} \delta^j u'(C_{t+j}) D_{t+j} \right] \text{ a.s.}$$

Unlike in equation (30), the right side of the previous equation depends on the agent's subjective consumption plans. From equation (18) follows that future consumption plans are of the form $C_{t+j}((P, D, W)^{t+j})$. Expected future consumption

⁴⁴ Obviously, the subsequent discussion takes for granted the fact that investors are homogeneous is not common knowledge.

⁴⁵ See Adam and Marcet (2011) for a discussion of how in the presence of trading constraints, this conclusion breaks down with linear consumption preferences.

thus depends on the agent's price expectations, say those implied by equation (26). Indeed, whatever are the agent's price expectations, subjective consumption plans will adjust, so as to satisfy the first-order condition (29). As a result, equation (31) will also hold. Equation (31) thus fails to impose any restriction on what optimizing agents can possibly believe about the price process, given their knowledge about the processes for W_t and D_t . With the considered nonlinear utility function, we can thus simultaneously assume that agents maximize utility, hold the subjective price beliefs (26)–(27) and have rational expectations about dividends and wages.

The fact that equilibrium prices can be written as a discounted sum of the form (31) differs notably from the standard pricing formula studied in modern asset-pricing theory, which determines equilibrium prices using

$$P_t u'(W_t + D_t) = E_t^{\mathcal{P}} \left[\sum_{j=1}^{\infty} \delta^j u'(W_{t+j} + D_{t+j}) D_{t+j} \right] \text{ a.s.}$$

The RHS of this equation uses equilibrium values of future consumption instead of the subjective future consumption plans showing up in equation (31). In this sense, stock prices satisfying (31) are not efficient, because they do not discount dividends with the *equilibrium* discount factor.

C. Learning about Capital Gains

Under internal rationality, the specification of the perceived price process (26) dictates the way that agents learn from observed prices. The presence of an unobserved permanent component in equation (26) gives rise to an optimal filtering problem. To obtain a parsimonious description of this problem, we specify conjugate prior beliefs about the unobserved persistent component $\ln \beta_t$ at $t = 0$. Specifically, agent i 's prior is

$$(32) \quad \ln \beta_0 \sim N(\ln m_0^i, \sigma^2),$$

where prior uncertainty σ^2 is assumed to be equal to its Kalman filter steady-state value, i.e.,

$$(33) \quad \sigma^2 \equiv \frac{-\sigma_\nu^2 + \sqrt{(\sigma_\nu^2)^2 + 4\sigma_\nu^2\sigma_\varepsilon^2}}{2}.$$

Equations (26), (27), and (32), and knowledge of the dividend and wage income processes (17) then jointly specify agents' probability beliefs \mathcal{P}^i .

The optimal Bayesian filter then implies that the posterior beliefs following some history ω^t are given by⁴⁶

$$(34) \quad \ln \beta_t | \omega^t \sim N(\ln m_t^i, \sigma^2),$$

⁴⁶See West and Harrison (1997, Theorem 3.1). Choosing a value for σ^2 different from the steady-state value (33) would only add a deterministically evolving variance component σ_t^2 to posterior beliefs with the property $\lim_{t \rightarrow \infty} \sigma_t^2 = \sigma^2$, i.e., it would converge to the steady-state value.

with

$$(35) \quad \ln m_t^i = \ln m_{t-1}^i - \frac{\sigma_v^2}{2} + g \left(\ln P_t - \ln P_{t-1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_{t-1}^i \right),$$

$$(36) \quad g = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

Agents' beliefs can thus be parsimoniously summarized by a single state variable (m_t^i) describing agents' degree of optimism about future capital gains. These beliefs evolve recursively according to equation (35) and imply that

$$(37) \quad \ln m_t^i = \ln E_t^{\mathcal{P}^i} \left[\frac{P_{t+1}}{P_t} \right] - \sigma^2/2,$$

so that equation (35) is—up to some (small) variance correction and the presence of a time lag—identical to the adaptive prediction model (13) considered in Section IID.

The subjective price beliefs (26), (27), and (32) generate perfect foresight equilibrium price expectations in the special case in which prior beliefs are centered at the growth rate of dividends, i.e.,

$$\ln m_0^i = \ln \beta^D,$$

and when considering the limiting case with vanishing uncertainty, where $(\sigma_\varepsilon^2, \sigma_\nu^2, \sigma_D^2, \sigma_W^2) \rightarrow 0$. Agents' prior beliefs at $t = 0$ about price growth in $t \geq 1$ then increasingly concentrate at the perfect foresight outcome $\ln \beta^D$ (see equations (26) and (27)). With price and dividend expectations being at their perfect foresight value, the perfect foresight equilibrium prices implied by Proposition 2 for the vanishing noise limit become the equilibrium outcome at $t = 0$. Importantly, it continues to be possible to study learning dynamics in the limit with vanishing risk: keeping the limiting ratio $\sigma_\nu^2 / \sigma_\varepsilon^2$ finite and bounded from zero as uncertainty vanishes, the Kalman gain parameter g , defined in equation (36), remains well specified and satisfies $\lim_{\sigma_\varepsilon^2 \rightarrow 0} \frac{\sigma_\nu^2}{\sigma_\varepsilon^2} = \lim_{\sigma_\varepsilon^2 \rightarrow 0} \frac{g^2}{1-g}$. We will exploit this fact in Section VII when presenting analytical results.

VI. Dynamics under Learning

This section explains how equilibrium prices are determined under the subjective beliefs introduced in the previous section and how they evolve over time.

Agents' stock demand is given by equation (22). Stock demand depends on the belief m_t^i , which characterizes agents' capital gains expectations. These beliefs evolve according to (35). As a benchmark, we shall now assume that all agents hold identical beliefs ($m_t^i = m_t$ for all i). In doing so, we assume that the marginal investor has return expectations in line with those documented for the survey data. We consider the homogeneous agent case first, so as to stay as close as possible to the representative agent settings typically considered under RE. Section VIIIB considers extensions to settings with heterogeneous beliefs.

Using the representative agent assumption, the fact that stocks are in unit supply and imposing market clearing in periods t and $t - 1$ in equation (22), one obtains that the equilibrium price in any period $t \geq 0$ solves

$$(38) \quad 1 = S\left(1, \frac{P_t}{D_t}, \frac{W_t}{D_t}, m_t\right).$$

The beliefs m_t and the price dividend ratio P_t/D_t are now simultaneously determined via equations (35) and (38). Due to a complementarity between realized capital gains and expected future capital gains,⁴⁷ this simultaneity can give rise to multiple market-clearing price and belief pairs. In the learning literature, the standard approach to resolve this issue consists of assuming that agents use only lagged endogenous variables to update model estimates (Eusepi and Preston 2011; Evans and Honkapohja 2001; Marcet and Nicolini 2003; Sargent, Williams, and Zha 2009). In our benchmark specification we shall follow this approach, but we provide below Bayesian foundations for lagged updating in a setting where agents can observe current prices. Furthermore, as we show in Section XC, lagged belief updating is objectively optimal because agents who update beliefs using current price information experience lower utility in equilibrium.

Introducing an updating lag into equation (35) can be justified as internally rational by slightly modifying the information structure. The modification is relatively straightforward and consists of assuming that agents observe at any time t information about the lagged transitory price growth component ε_{t-1} entering equation (26). In such a setting, it is optimal to give less weight to the last available price growth observation, when estimating the permanent component β_t . This is so because the last observation is more noisy. Formally, online Appendix A7 shows that in the limit where agents learn all transitory factors with a lag, Bayesian updating implies

$$(39) \quad \ln m_t = \ln m_{t-1} + g(\ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1}) + g \ln \varepsilon_t^1,$$

where updating now occurs using only lagged price growth (even though agents do observe current prices) and where $\ln \varepsilon_t^1 \sim iN\left(\frac{-\sigma_\varepsilon^2}{2}, \sigma_\varepsilon^2\right)$ is a time t innovation to agent's information set (unpredictable using information available to agents up to period $t - 1$). The shock $\ln \varepsilon_t^1$ thereby captures the information that agents receive in period t about the transitory price growth component $\ln \varepsilon_{t-1}$.

With this slight modification, agents' beliefs m_t are now predetermined at time t , so that the economy evolves according to a uniquely determined recursive process: equation (38) determines the market-clearing price for period t given the beliefs m_t and equation (39) determines how time t beliefs are updated following the observation of the new market-clearing price.⁴⁸

⁴⁷ Intuitively, a higher PD ratio implies higher realized capital gains and thus higher expectations of future gains via equation (35). Higher expected future gains may in turn induce a higher willingness to pay for the asset, thereby justifying the higher initial PD ratio.

⁴⁸ There could still be an indeterminacy arising from the fact that $S(\cdot)$ is nonlinear, so that equation (38) may not have a unique solution. We have not encountered such problems in our analytical solution or when numerically solving the model.

VII. Equilibrium: Analytic Findings

This section derives a closed-form solution for the equilibrium asset price for the special case where all agents hold the same subjective beliefs \mathcal{P} and where these beliefs imply no (or vanishing) uncertainty about future prices, dividends, and wages. While the absence of uncertainty is unrealistic from an empirical standpoint, it helps us in deriving key insights into how the equilibrium price depends on agents' beliefs, as well as on how prices and beliefs evolve over time.⁴⁹ The empirically more relevant case with uncertainty will be considered in Section IX using numerical solutions.

We present a series of results that increasingly adds assumptions on agents' beliefs system \mathcal{P} . The next section provides a closed-form expression for the equilibrium PD ratio as a function of agents' subjective expectations about future stock market returns for any belief system \mathcal{P} without uncertainty. Section VIIB then discusses the pricing implications of this result for the subjective capital gains beliefs presented in Section V. Finally, Section VIIC shows how the interaction between asset price behavior and subjective belief revisions can temporarily delink asset prices from their fundamental value, i.e., give rise to a self-reinforcing boom and bust cycle in asset prices along which subjective expected returns rise and fall.

A. Main Result

Letting $R_{t+1} \equiv (P_{t+1} + D_{t+1})/P_t$ denote the stock return, we have following main result.⁵⁰

PROPOSITION 3: *Suppose $u(C) = C^{1-\gamma}/(1-\gamma)$, agents' beliefs \mathcal{P} imply no uncertainty about future prices, dividends, and wages. Assume given posterior beliefs about prices, dividends and wages that satisfy*

$$(40) \quad \lim_{T \rightarrow \infty} E_t^{\mathcal{P}} R_T > 1 \quad \text{and} \quad \lim_{T \rightarrow \infty} E_t^{\mathcal{P}} \left(\sum_{j=1}^T \left(\prod_{i=1}^j \frac{1}{R_{t+i}} \right) W_{t+j} \right) < \infty,$$

then, if $|\underline{S}|, |\bar{S}|$ are sufficiently large, the equilibrium PD ratio in period t is given by

$$(41) \quad \begin{aligned} \frac{P_t}{D_t} = & \left(1 + \frac{W_t}{D_t} \right) \sum_{j=1}^{\infty} \left((\delta^{\frac{1}{\gamma}})^j \left(E_t^{\mathcal{P}} \prod_{i=1}^j \frac{1}{R_{t+i}} \right)^{1-\frac{1}{\gamma}} \right. \\ & \left. - \frac{1}{D_t} E_t^{\mathcal{P}} \left(\sum_{j=1}^{\infty} \left(\prod_{i=1}^j \frac{1}{R_{t+i}} \right) W_{t+j} \right) \right). \end{aligned}$$

⁴⁹ An analytic solution can be found because in the absence of uncertainty one can evaluate more easily the expectations of nonlinear functions of future variables showing up in agents' FOCs.

⁵⁰ The proof can be found in online Appendix A8.

Conditions (40) insure that the infinite sums in the pricing equation (41) converge.⁵¹ Under the additional assumption that agents hold rational wage and dividend expectations and that $W_t/D_t = \rho$, equation (41) simplifies further to⁵²

$$(42) \quad \frac{P_t}{D_t} = (1 + \rho) \sum_{j=1}^{\infty} \left((\delta^{\frac{1}{\gamma}})^j \left(E_t^{\mathcal{P}} \prod_{i=1}^j \frac{1}{R_{t+i}} \right)^{\frac{\gamma-1}{\gamma}} \right) - \rho \left(\sum_{j=1}^{\infty} (\beta^D)^j \left(E_t^{\mathcal{P}} \prod_{i=1}^j \frac{1}{R_{t+i}} \right) \right).$$

We now discuss the implications of equation (42), focusing on the empirically relevant case where $\rho > 0$ and $\gamma > 1$.

Consider first the upper term on the RHS of equation (42), which is decreasing in the expected asset returns. This emerges because for $\gamma > 1$ the wealth effect of a change in return expectations then dominates the substitution effect, so that expected asset demand and therefore the asset price has a tendency to decrease as return expectations increase. The negative wealth effect thereby increases in strength if the ratio of wage to dividend income (ρ) increases. This is the case because higher return expectations also reduce the present value of wage income.

Next, consider the lower term on the RHS of equation (42), including the negative sign premultiplying it. This term depends positively on the expected returns and captures a substitution effect that is associated with increased return expectations. This substitution effect only exists if $\rho > 0$, i.e., only in the presence of nondividend income, and it is increasing in ρ . It implies that increased return expectations are associated with increased stock demand and thus with a higher *PD* ratio in equilibrium. It is this term that allows the model to match the positive correlation between expected returns and the *PD* ratio.

This substitution effect is present even in the limiting case with log consumption utility ($\gamma \rightarrow 1$). The upper term on the RHS of equation (42) then vanishes because the substitution and wealth effects associated with changes in expected returns cancel each other, but the lower term still induces a positive relationship between prices and return expectations. The substitution effect is also present for $\gamma > 1$ and can then dominate the negative wealth effect arising from the upper term on the RHS of (42). Consider, for example, the opposite limit with $\gamma \rightarrow \infty$. Equation (42) then delivers

$$\frac{P_t}{D_t} = \sum_{j=1}^{\infty} \left(1 + \rho \sum_{j=1}^{\infty} (1 - (\beta^D)^j) \right) \left(E_t^{\mathcal{P}} \prod_{i=1}^j \frac{1}{R_{t+i}} \right).$$

Since $\beta^D > 1$, there is a positive relationship between prices and expected asset returns, whenever ρ is sufficiently large. The two limiting results ($\gamma \rightarrow 1$ and

⁵¹ These are satisfied, for example, for the expectations associated with the perfect foresight RE solution. Equation (41) then implies for $\rho = 1$ that the *PD* ratio equals the perfect foresight *PD* ratio, i.e., the value given by equation (24) with $\varepsilon_t^W = 1$, as is easily verified. Conditions (40) are equally satisfied for the subjective beliefs defined in Section V, when considering the case with vanishing uncertainty ($\sigma_\varepsilon^2, \sigma_\nu^2, \sigma_D^2, \sigma_W^2 \rightarrow 0$).

⁵² In deriving equation (42) we abstract from transitional dynamics in W_t/D_t and set $W_t/D_t = \rho$.

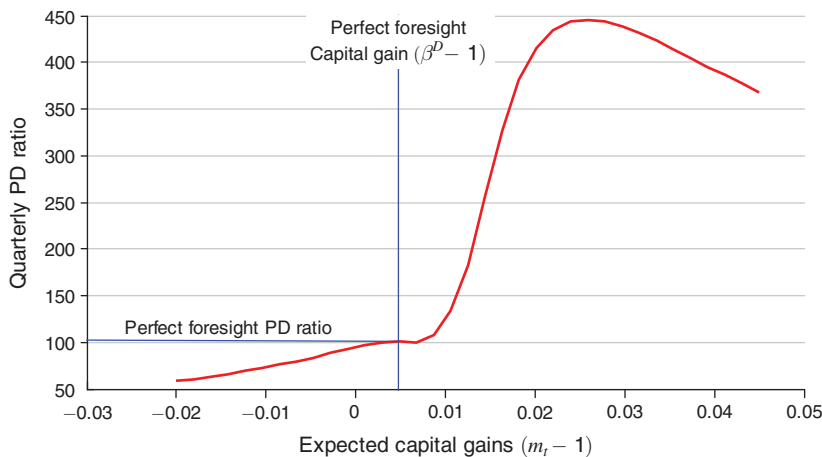


FIGURE 5. QUARTERLY PD RATIO AND EXPECTED CAPITAL GAINS
(Vanishing Noise Limit, Estimated Model from Table 3, Diagonal Matrix)

$\gamma \rightarrow \infty$) thus suggest that for sufficiently large ρ the model can generate a positive relationship between return expectations and the *PD* ratio, in line with the evidence obtained from survey data.

B. *PD* Ratio and Expected Capital Gains

Since future stock returns depend on the belief system and on current *PD* ratio, equation (42) does not yet give an explicit solution for the *PD* ratio. For the belief system introduced in Section V,⁵³ we have $E_t^P[P_{t+i}] = (m_t)^i P_t$ and $E_t^P D_{t+i} = (\beta^D)^i D_t$, so that without uncertainty

$$E_t^P R_{t+i}^{-1} = \frac{E_t^P P_{t+i-1}}{E_t^P P_{t+i} + E_t^P D_{t+i}} = \left(m_t + (\beta^D/m_t)^{i-1} \beta^D \frac{D_t}{P_t} \right)^{-1}.$$

Substituting this into (42) gives a nonlinear relationship between the *PD* ratio and the subjective capital gain expectations m_t . Since a closed-form solution for the *PD* ratio is unavailable, we show numerical solutions of this equation.

Figure 5 depicts the relationship between the *PD* ratio and m_t using the parameterization employed in our quantitative application later on, but abstracting from future uncertainty.⁵⁴ Figure 5 shows that there is a range of price growth beliefs around the perfect foresight value ($m_t = \beta^D$) over which the *PD* ratio depends positively on expected price growth, similar to the positive relationship between expected returns and the *PD* ratio derived analytically in the previous section. Over this range, the substitution effect dominates the wealth effect because our calibration implies that dividend income finances only a small share of total consumption (approximately 4.3 percent). As a result, stock market wealth is only a small share

⁵³ Online Appendix A9 proves that condition (40) is satisfied for all beliefs $m_t > 0$.

⁵⁴ The parameter values are given by those listed in Table 2 and by the estimated parameters for the model with diagonal matrix from Table 3, i.e., $g = 0.262$, $\gamma = 2.03$, $\delta = 0.99514$, $p = 0.95$.

TABLE 2—MODEL CALIBRATION

| Parameter | Value | Calibration target |
|----------------------|-----------------------|--|
| β^D | 1.0048 | Average quarterly real dividend growth |
| σ_D | 0.0192 | Standard deviation quarterly real dividend growth |
| ρ | 22 | Average consumption-dividend ratio |
| σ_{DW} | $-3.51 \cdot 10^{-4}$ | Jointly chosen such that $\text{corr}(C_{t+1}/C_t, D_{t+1}/D_t) = 0.2$ |
| σ_W | 0.0186 | and $\text{std}(C_{t+1}/C_t) = (1/7) \text{std}(D_{t+1}/D_t)$ |
| σ_ε | 0.0816 | SD of quarterly real stock price growth |

of the total present value of household wealth (the same 4.3 percent) when beliefs assume their perfect foresight value ($m_t = \beta^D$).

Figure 5 also reveals that there exists a capital gains belief beyond which the PD ratio starts to decrease for higher m . Mathematically, this occurs because if $m_t \rightarrow \infty$, expected returns also increase without bound,⁵⁵ so that $E_t^P \prod_{i=1}^j \frac{1}{R_{t+i}} \rightarrow 0$. From equation (42) one then obtains $\frac{P_t}{D_t} \rightarrow 0$.

The economic intuition for the existence of a maximum PD ratio is as follows: for higher m_t the present value of wage income is declining, as increased price growth optimism implies higher expected returns⁵⁶ and therefore a lower discount factor. This can be seen by noting that the FOC (29) can alternatively be written as

$$1 = \delta E_t^P \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right],$$

which implies that increased return expectations $E_t^P R_{t+1}$ imply a lower discount factor $\delta E_t^P [(C_{t+1}/C_t)^{-\gamma}]$.⁵⁷ With increased optimism, the present value of wage income thus falls. At the same time, stock market wealth initially increases strongly. Indeed, at the maximum PD ratio, stock market wealth amounts to approximately 4.5 times the value it assumes in the perfect foresight solution (see Figure 5). This relative wealth shift has the same effect as a decrease in the wage to nonwage income ratio ρ . As argued in Section VIIA, for sufficiently small values of ρ the income effect starts to dominate the substitution effect, so that prices start to react negatively to increased return optimism.

C. Endogenous Boom and Bust Dynamics

We now explain how the interplay between price realizations and belief updating can temporarily delink asset prices from their fundamental values. This process emerges endogenously and takes the form of a sustained asset price boom along which expected returns rise and that ultimately results in a price bust along which expected returns fall. This feature allows the model to generate volatile asset prices and to capture the positive correlation between expected returns and the PD ratio.

⁵⁵This follows from $E_t^P R_{t+i+1} = E_t^P \frac{P_{t+i+1} + D_{t+i+1}}{P_{t+i}} > E_t^P \frac{P_{t+i+1}}{P_{t+i}} = m_t$.

⁵⁶This is shown in online Appendix A11, which depicts the relationship between expected capital gains and expected returns at various forecast horizons.

⁵⁷This holds true under the maintained assumption of no or vanishing uncertainty.

Consider Figure 5 and a situation in which agents become optimistic, in the sense that their capital gains expectations m_t increase slightly above the perfect foresight value $m_{t-1} = \beta^D$ entertained in the previous period.⁵⁸ Figure 5 shows that this increase in expectations leads to an increase in the PD ratio, i.e., $P_t/D_t > P_{t-1}/D_{t-1}$. Moreover, due to the relatively steep slope of the PD function, realized capital gains will strongly exceed the initial increase in expected capital gains. The belief updating equation (39) then implies further upward revisions in price growth expectations and thus further capital gains, leading to a sustained asset price boom in which the PD ratio and return expectations jointly move upward.

The price boom comes to an end when expected price growth reaches a level close to where the PD function in Figure 5 reaches its maximum.⁵⁹ At this point, stock prices grow at most at the rate of dividends (β^D), but agents hold considerably more optimistic expectations about future capital gains ($m_t > \beta^D$). Investors' high expectations will thus be disappointed, which subsequently leads to a reversal. During a price boom, expected price growth and actual price growth thus mutually reinforce each other. The presence of an upper bound in prices implies, however, that the boom must come to an end. Price growth must thus eventually become very low, sending actual and expected stock price growth eventually down.

The previous dynamics are also present in the stochastic model considered in the next sections. They introduce low frequency movements in the PD ratio, allowing the model to replicate boom and bust dynamics and thereby empirically plausible amounts of asset price volatility, despite assuming standard consumption preferences. These dynamics also generate a positive correlation between the PD ratio and expected returns.⁶⁰

In the deterministic model, however, the dynamics for the PD ratio tend to be temporary phenomena because beliefs tend to converge to the perfect foresight equilibrium.⁶¹

LEMMA 1: *Consider the limiting case without uncertainty and suppose investors hold rational dividend and wage beliefs:*

- (i) For any $m_t > 0$, we have $\lim_{t \rightarrow \infty} m_t = \beta^D$, whenever $\lim_{t \rightarrow \infty} m_t$ exists.
- (ii) For m_t sufficiently close to β^D and $g < \frac{1}{2}$, we have $\lim_{t \rightarrow \infty} m_t = \beta^D$ if

$$(43) \quad -1 < \frac{\beta^D}{PD(\beta^D)} \frac{\partial PD(m)}{\partial m} \Big|_{m=\beta^D} < 1,$$

⁵⁸ In the model with uncertainty, such upward revisions can be triggered by fundamentals, e.g., by an exceptionally high dividend growth realization in the previous period, which is associated with an exceptionally high price growth realization.

⁵⁹ In the model with noise, fundamental shocks, e.g., a low dividend growth realization, can cause the process to end well before reaching this point.

⁶⁰ While the arguments above only show that expected capital gains correlate positively with the PD ratio, online Appendix A11 shows that expected capital gains and expected returns comove positively, so that expected returns also comove positively with the PD ratio.

⁶¹ The proof of Lemma 1 can be found in online Appendix A10.

TABLE 3—ASSET-PRICING MOMENTS, DATA, AND ESTIMATED MODELS

| | US data 1946:I–2012:I (quarterly real values) | Subjective belief model (efficient Σ matrix) | | Subjective belief model (diagonal Σ matrix) | |
|--------------------------------------|---|--|-----------------|--|-----------------|
| | Moment | Moment | <i>t</i> -ratio | Moment | <i>t</i> -ratio |
| $E[PD]$ | 139.8 | 120.1 | −0.79 | 115.2 | −0.98 |
| $\text{std}[PD]$ | 65.2 | 95.6 | 2.06 | 88.2 | 1.56 |
| $\text{corr}[PD_t, PD_{t-1}]$ | 0.98 | 0.98 | 0.65 | 0.98 | 0.90 |
| $\text{std}[R]$ | 8.00 | 8.88 | 2.20 | 7.74 | −0.64 |
| \mathbf{c} | −0.0041 | −0.0055 | −1.11 | −0.0050 | −0.77 |
| R^2 | 0.25 | 0.23 | −0.13 | 0.20 | −0.34 |
| $E[R] - 1$ | 1.89 | 1.91 | 0.04 | 1.82 | −0.15 |
| $E[R^b] - 1$ | 0.13 | 1.01 | 5.16 | 0.99 | 5.08 |
| <i>UBS survey data</i> | | | | | |
| $\text{corr}[PD_t, E_t^P R_{t,t+4}]$ | 0.79 | 0.80 | 0.07 | 0.79 | 0.01 |
| <i>Estimates</i> | | | | | |
| $\hat{\xi}$ | | 0.0282 | | 0.0262 | |
| $\hat{\delta}$ | | 0.99524 | | 0.99514 | |
| $\hat{\gamma}$ | | 2.05 | | 2.03 | |
| \hat{p} | | 0.95 | | 0.95 | |

Notes: The table reports US asset-pricing moments (column 2) using the data sources described in online Appendix A1, the moments of the estimated models (columns 3 and 5) and *t*-ratios for the model moments (columns 4 and 6) from the SMM estimation. *t*-ratios are defined as (data moment − model moment)/(estimated standard deviation of the model moment). The reported moments are as follows: $E[PD]$, $\text{std}(PD)$ and $\text{corr}[PD_t, PD_{t-1}]$ denote the mean, standard deviation, and autocorrelation of the quarterly price dividend ratio, respectively; $E[R]$ and $\text{std}[R]$ denote the mean and standard deviation of the gross real quarterly stock return, expressed in percentage points, respectively; $E[R^b]$ is the gross mean risk-free interest rate, expressed in percentage points; \mathbf{c} and R^2 denote, respectively, the regression coefficient and R^2 value obtained in equation (5) when $R_{t,t+N}$ is excess stock returns and $N = 5$ years; $\text{corr}[PD_t, E_t^P R_{t,t+4}]$ denotes the correlation between the *PD* ratio and the subjective return expectations, with these being measured in the data column using the mean of the UBS survey (own portfolio), deflated by the mean inflation expectations from the SPF survey. The estimated parameters are the updating gain g from equation (39), the time discount factor δ , the coefficient of relative risk aversion γ and the persistence p of the wage-dividend ratio from equation (16).

where $PD(m)$ is the equilibrium *PD* ratio associated with beliefs m , as implied by equation (42).

The first result in the lemma provides a global convergence result. It shows that if beliefs settle down in this economy, they must settle down on the perfect foresight equilibrium value. When this is the case, equilibrium prices also converge to the perfect foresight value. While technically one cannot rule out convergence to deterministic or chaotic cycles, the second result in the lemma shows that locally beliefs do converge to the perfect foresight equilibrium, whenever the elasticity of the *PD* ratio with respect to price growth beliefs is below 1 in absolute value and the gain parameter g not too large. Condition (43) is satisfied, for example, for the parameterization of the estimated models reported in Table 3.

To illustrate the global belief dynamics further, Figure 6 depicts how beliefs evolve over time using the parameterization of the estimated model from Table 3.⁶² The arrows in the figure indicate, starting from any point (m_t, m_{t-1}) in the plane,

⁶²The vector field uses the parameters from the estimated model with diagonal matrix and estimated p reported in Table 3. The vector fields for the other models from Table 3 look almost identical.

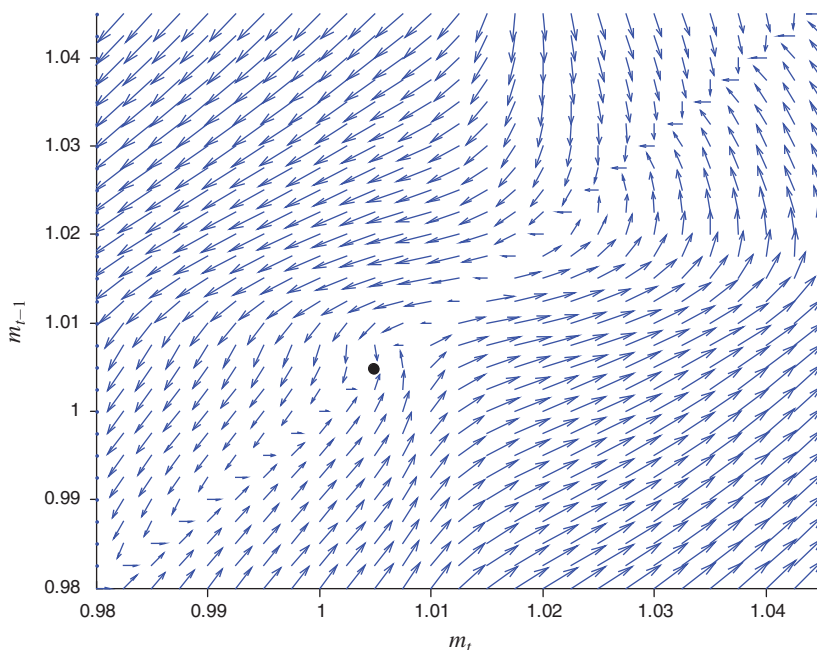


FIGURE 6. ILLUSTRATION OF GLOBAL BELIEF DYNAMICS UNDER LEARNING IN THE ABSENCE OF SHOCKS (*Estimated Model from Table 3, Diagonal Matrix*)

Note: The black dot indicates the perfect foresight equilibrium belief.

the direction in which the belief pair (m_t, m_{t-1}) evolves.⁶³ The black dot indicates the position of the perfect foresight equilibrium $(m_t = m_{t-1} = \beta^D)$, which is a rest point of the dynamics. In line with what we find when simulating the model, Figure 6 strongly suggests that beliefs globally converge to the perfect foresight equilibrium in the absence of stochastic disturbances.

Figure 6 also shows that it takes time for beliefs to settle down at the perfect foresight equilibrium and that agents will make persistent forecast errors along the transition path. In particular, when agents are optimistic and stock prices high ($m_t > \beta^D$), there is a tendency for prices and beliefs to return toward their perfect foresight values, so that realized capital gains are low. Similarly, when agents are pessimistic and stock prices low ($m_t < \beta^D$), the tendency of prices and beliefs to return to the perfect foresight values implies that realized capital gains are high. Note that this pattern of forecast errors is consistent with the one documented for the survey data in Section II.

VIII. Matching Asset-Pricing Moments

This section evaluates the ability of the model to replicate key asset-pricing moments when using dividend and wage shocks as fundamental driving forces.

⁶³To increase readability of the graph, the length of the arrows l is nonlinearly rescaled by dividing by $l^{3/4}$ and then linearly adjusted, so as to fit into the picture.

The set of data moments that we seek to replicate is listed in the second column of Table 3. The first eight asset-pricing moments listed in the table are standardly used in the asset-pricing literature to summarize the main features of stock price volatility.⁶⁴ They include the mean, standard deviation, and autocorrelation of the quarterly PD ratio ($E[PD]$, $\text{std}[PD]$, and $\text{corr}[PD_t, PD_{t-1}]$, respectively), the mean and standard deviation of quarterly real stock returns ($E[r^s]$ and $\text{std}[r^s]$), the risk-free interest rate ($E[r^b]$) and the regression coefficient (\mathbf{c}) and R -squared value (R^2) obtained in equation (5) when $R_{t,t+N}$ is the five year ahead excess return.⁶⁵ As is well known, it is difficult for RE models with time separable utility functions to jointly match these moments.

We augment the standard set of moments in Table 3 by the correlation between the PD ratio and expected stock returns from the UBS survey, denoted $\text{corr}[PD_t, E_t^P R_{t+1}]$, so as to capture the behavior of survey data.

Numerically solving the nonlinear asset-pricing model with subjective beliefs turns out to be computationally time-consuming, despite the fact that we extensively rely on parallelization in the solution algorithms.⁶⁶ For this reason, we match the model to the data by calibrating most parameters and estimate only a key subset using the simulated method of moments (SMM).

Table 2 reports the calibrated parameters and the calibration targets.⁶⁷ The mean and standard deviation of dividend growth (β^D and σ_D) are chosen to match the corresponding empirical moments of the US dividend process. The ratio of nondividend to dividend income (ρ) is chosen to match the average dividend-consumption ratio in the United States for 1946–2011.⁶⁸ The standard deviation of wage innovations (σ_w) and the covariance between wage and dividend innovations (σ_{DW}) are chosen, in line with Campbell and Cochrane (1999), such that the correlation between consumption and dividend growth is 0.2 and the standard deviation of consumption growth is one-seventh of the standard deviation of dividend growth.⁶⁹ The perceived uncertainty in stock price growth (σ_ϵ) is set equal to the empirical standard deviation of stock price growth.⁷⁰

This leaves us with four remaining parameters: the updating parameter g , the time discount factor δ , the risk aversion parameter γ , and the persistence parameter for the wage-dividend ratio p . Letting $\theta = (g, \delta, \gamma, p)$ denote this set of parameters and Θ the set of admissible values, the SMM estimate $\hat{\theta}$ is given by

$$(44) \quad \hat{\theta} = \arg \min_{\theta \in \Theta} [\hat{S} - \mathcal{S}(\theta)]' \hat{\Sigma} [\hat{S} - \mathcal{S}(\theta)],$$

⁶⁴ These moments are also considered in Adam, Marcet, and Nicolini (2016).

⁶⁵ The regression also contains a constant whose value is statistically insignificant and not reported in the table.

⁶⁶ The numerical solution is obtained by numerically determining the stock demand function (22) solving the FOC (29) under the subjectively perceived dividend, wage, and price dynamics, where agents understand that their beliefs evolve according to (39). We verify that in the limiting case without uncertainty, our numerical solution algorithm recovers the analytical solution derived in Proposition 3. Furthermore, in the case with uncertainty, we insure the accuracy of the numerical solution by verifying that the Euler equation errors are in the order of 10^{-5} over the relevant area of the state space. Insuring this requires a considerable amount of adjustment by hand of the grid points and grid size used for spanning the model's state space. Further details of the solution approach are described in online Appendix A12. The MatLab code used for solving the model is available upon request.

⁶⁷ The targets are chosen to match features of the fundamental processes emphasized in the asset-pricing literature.

⁶⁸ See online Appendix A4 for further details.

⁶⁹ For details on how this can be achieved, see online Appendix A4.

⁷⁰ Since the gain parameter g will be small, σ_v^2 is also small, hence the contribution of σ_v^2 to the total variance in (26) is negligible.

where $\hat{\mathcal{S}}$ is the set of moments in the data to be matched (the ones listed in Table 3), $\mathcal{S}(\theta)$ the corresponding moment from the model for parameter values θ , and $\hat{\Sigma}$ a weighting matrix.

We pursue two estimation approaches, one that implements efficient SMM and an alternative one that emphasizes more directly the replication of the data moments. Both approaches have advantages and disadvantages, as we discuss below, but ultimately deliver similar estimates and model moments. All estimations exclude the risk-free rate from the set of moments to be matched, as the model has a hard time in fully replicating the equity premium. We thus only report the risk-free interest rate implied by the estimated models.

Our first estimation approach chooses the weighting matrix $\hat{\Sigma}$ in equation (44) to be equal to the inverse of the estimated covariance matrix of the data moments $\hat{\mathcal{S}}$, as required for efficient SMM estimation. While an efficient weighting matrix is desirable for estimation, it also causes some difficulties. First, the weighting matrix turns out to be approximately singular: following Adam, Marcet, and Nicolini (2016), we thus exclude some moments from the estimation that are nearly redundant, namely we exclude the excess return regression coefficient (c) from the set of estimated moments.⁷¹ Second, we also exclude the correlation between the *PD* ratio and surveyed expected returns, since the short sample for surveys provides a less reliable estimate of $\hat{\Sigma}$.

The second estimation approach uses a diagonal weighting matrix $\hat{\Sigma}$ in equation (44), with the diagonal entries consisting of the inverse of the individually estimated variances of the corresponding data moments in $\hat{\mathcal{S}}$. Although this alternative is less efficient from an econometric point of view, the matrix $\hat{\Sigma}$ is now guaranteed to be invertible, therefore we can use the full set of moments in the estimation.⁷² Furthermore, this criterion just minimizes the sum of *t*-statistics of all the moments, therefore emphasizes more directly matching the moments, as typically pursued in the asset-pricing literature.

An unconstrained minimization of the objective function (44) over $\theta = (g, \delta, \gamma, p)$ turns out to be numerically unstable and computationally too costly. For this reason, we impose additional restrictions on the parameter space Θ . These restrictions can only constrain the empirical performance of the model, so that the goodness of fit results presented below constitute a lower bound on what the model can potentially achieve. Specifically, we impose $\delta(\beta^D)^{-\gamma} = 0.995(\beta^D)^{-2}$, where β^D assumes the value from Table 3. This additional restriction is inspired by the fact that—according to our experience—the model can perform reasonably well for $(\delta, \gamma) = (0.995, 2)$ and helps resolving numerical instabilities in our solution routines. We furthermore restrict the persistence parameter to $p \in \{0.95, 0.999\}$, which is inspired by the fact that the sample autocorrelation of $\log(1 + W_t/D_t)$ is very high in US postwar data (about 0.99).

⁷¹ See Section V and the online Appendix in Adam, Marcet, and Nicolini (2016) for details on asymptotic distribution of SMM, how to use a systematic criterion for excluding moments, details on how to estimate $\hat{\Sigma}$ and how to compute $\mathcal{S}(\theta)$ using small samples.

⁷² As mentioned before, the risk-free rate is always excluded from estimation.

Table 3 reports the estimation outcome in terms of implied model moments, estimated parameters, and t -ratios.⁷³

In terms of estimated parameters, the discount factor is estimated to be close to 1 and relative risk aversion is slightly above 2. The estimated gains (\hat{g}) are very close to the values obtained in the empirical Section IID.

For both estimation approaches, the model matches the data moments rather well. In particular, the model easily replicates the positive correlation between the PD ratio and expected stock returns ($\text{corr}[PD_t, E_t^P R_{t+1}]$), in line with the value found in the survey data. This is achieved, even though this moment was not used in the estimation using the efficient weighting matrix. The model also performs well in terms of producing sufficient volatility for the PD ratio ($\text{std}[PD]$) and stock returns ($\text{std}[r^s]$). If anything, the model tends to produce too much volatility. The model also succeeds in replicating the mean and autocorrelation of the PD ratio ($E[PD]$ and $\text{corr}[PD_t, PD_{t-1}]$) and the evidence on excess return predictability (c and R^2), even though the regression coefficient c was not included in the set of moments to be matched.

The most significant shortcoming of the model concerns the equity premium. While it matches the average stock return ($E[r^s]$), it predicts a too high value for the risk-free rate ($E[r^b]$). Nevertheless, the model can explain about one-half of the equity premium observed in the data. Given the low estimated value for the degree of relative risk aversion ($\hat{\gamma}$), this is remarkable and we explore the mechanism behind this outcome further in Section VIIIB below.

Abstracting from the mean risk-free rate, the estimation using the diagonal weighting matrix generates t -ratios below 2 for all model moments and often even t -ratios below 1. The estimation using the efficient weighting matrix performs similarly well, but implies a slightly high volatility for stock returns and the PD ratio. The empirical performance of the subjective belief model is overall very good, especially when compared to the performance of the rational expectations version of the model, as reported in Table 4 using the same parameters as for the estimated model (with diagonal matrix) in Table 3. Under RE, the t -ratios all increase in absolute terms, with some of the increases being quite dramatic. With objective price beliefs, the model produces insufficient asset price volatility (too low values for $\text{std}[PD]$ and $\text{std}[r^s]$) and the wrong sign for the correlation between the PD ratio and expected stock returns ($\text{corr}[PD_t, E_t^P R_{t+1}]$). It also gives rise to a small negative equity premium.⁷⁴ These features are rather robust across alternative parameterizations of the RE model and highlight the strong quantitative improvement obtained by incorporating subjective belief dynamics.

Table 5 shows that the performance of the subjective belief model is also rather robust across different values for the persistence parameter p of the wage-dividend process. The table reports the estimation outcomes when repeating the estimation

⁷³The t -ratio is the ratio of the gap between the model and the data moment over the standard deviation of the moment in the data, as implied by the weighting matrix. For excluded moments we use the individually estimated standard deviations in the numerator. For the case of an efficient weighting matrix, the t -ratio for moments included is computed according to the proper covariance matrix that delivers a standard normal distribution (see the online Appendix in Adam, Marcet, and Nicolini 2016).

⁷⁴For the considered parameterization, the equity premium is slightly negative because stocks are a hedge against wage income risk.

TABLE 4—RE ASSET-PRICING MOMENTS

| | US data | RE model | |
|--------------------------------------|---------|----------|-----------------|
| | Moment | Moment | <i>t</i> -ratio |
| $E[PD]$ | 139.8 | 108.3 | −1.26 |
| $\text{std}[PD]$ | 65.2 | 11.3 | −3.64 |
| $\text{corr}[PD_t, PD_{t-1}]$ | 0.98 | 0.95 | −11.20 |
| $\text{std}[R]$ | 8.00 | 1.49 | −16.22 |
| \mathbf{c} | −0.0041 | −0.0181 | −11.09 |
| R^2 | 0.25 | 0.72 | 3.66 |
| $E[R] - 1$ | 1.89 | 1.41 | −1.04 |
| $E[R^b] - 1$ | 0.13 | 1.52 | 8.15 |
| <i>UBS survey data</i> | | | |
| $\text{corr}[PD_t, E_t^P R_{t,t+4}]$ | 0.79 | −1.00 | −26.36 |

Notes: The table reports US asset-pricing moments (second column) using the data sources described in online Appendix A1, the moments of the rational expectations (RE) model (column 3), and the *t*-ratios of the RE model (column 4). The RE model uses the parameterization of the estimated subjective belief model from Table 3 (diagonal matrix). See Table 3 for a description of the moment labels in the first column.

TABLE 5—ASSET-PRICING MOMENTS (*Restricted Estimation*)

| Restriction imposed | US data 1946:I–2012:I (quart. real values) | Subjective belief model (diagonal matrix) $p = 0$ | | Subjective belief model (diagonal matrix) $p = 0.999$ | |
|--------------------------------------|--|--|-----------------|--|-----------------|
| | Moment | Moment | <i>t</i> -ratio | Moment | <i>t</i> -ratio |
| $E[PD]$ | 139.8 | 112.9 | −1.07 | 116.8 | −0.92 |
| $\text{std}[PD]$ | 65.2 | 82.1 | 1.14 | 98.5 | 2.25 |
| $\text{corr}[PD_t, PD_{t-1}]$ | 0.98 | 0.98 | −0.22 | 0.99 | 1.78 |
| $\text{std}[R]$ | 8.00 | 7.91 | −0.22 | 7.73 | −0.68 |
| \mathbf{c} | −0.0041 | −0.0045 | −0.36 | −0.0037 | 0.27 |
| R^2 | 0.25 | 0.15 | −0.74 | 0.14 | −0.79 |
| $E[R] - 1$ | 1.89 | 1.83 | −0.14 | 1.81 | −0.17 |
| $E[R^b] - 1$ | 0.13 | 0.97 | 4.92 | 1.00 | 5.12 |
| <i>UBS survey data</i> | | | | | |
| $\text{corr}[PD_t, E_t^P R_{t,t+4}]$ | 0.79 | 0.85 | 0.82 | 0.75 | −0.65 |
| <i>Estimates</i> | | | | | |
| \hat{g} | | 0.0222 | | 0.0272 | |
| $\hat{\delta}$ | | 0.99510 | | 0.99519 | |
| $\hat{\gamma}$ | | 2.02 | | 2.04 | |

Notes: The table reports US asset-pricing moments (column 2) using the data sources described in online Appendix A1, the moments and *t*-ratios of the estimated subjective belief model when restricting the persistence parameter p from equation (16) to zero (columns 3 and 4) and when restricting p to 0.999 (columns 5 and 6). See Table 3 for a description of the labels used in the first column.

from Table 3 but imposing $p = 0$ (no persistence) and $p = 0.999$ (near unit root behavior). The ability of the subjective belief model to match the empirical asset-pricing moments is hardly affected by the persistence parameter.

Figure 7 illustrates how the subjective belief model improves empirical performance. The figure depicts the equilibrium PD ratio (y-axis) as a function of agents' capital gain beliefs (x-axis). It graphs this relationship once for the model with uncertainty (dark line) and once for the vanishing noise limit analyzed in the previous

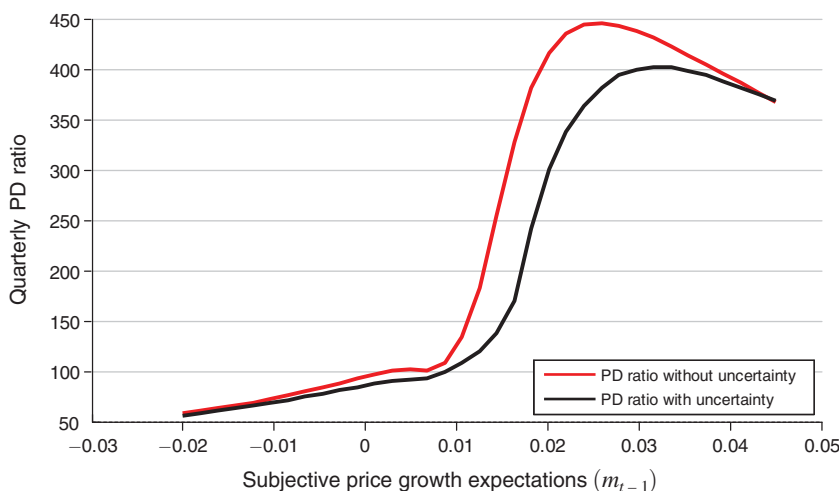


FIGURE 7. THE EFFECTS OF UNCERTAINTY ON THE EQUILIBRIUM PD RATIO
(Estimated Model from Table 3, Diagonal Matrix)

section (lighter line).⁷⁵ While the presence of price, dividend and wage risk lowers the equilibrium *PD* ratio compared to a setting without risk, the functional form of the relationship remains qualitatively unchanged. The intuition from the vanishing noise limit thus carries over to the model with noise: the model continues to give rise to occasional boom and bust dynamics in asset prices.

A. Further Evidence on Model Performance

Table 6 presents further asset-pricing moments that have not been used in the estimation of the model. It reports data moments, the corresponding model moments for the estimated subjective belief model from Table 3, for the RE version of model and for an augmented subjective belief model featuring stock supply shocks, as introduced below.

Table 6 shows that the subjective belief model successfully replicates the low volatility of the risk-free interest rate present in the data. The subjective belief model even slightly underpredicts volatility. This shows that the ability to generate highly volatile stock returns does not rely on counterfactually making the risk-free interest rate very volatile. In terms of matching the low volatility of the risk-free rate, the performance of the subjective belief model is approximately as good as that of the RE model.

Table 6 also presents evidence on the quarterly autocorrelations of stock returns and excess stock returns. It shows that the subjective belief model overpredicts both autocorrelations relative to the ones present in the data. As Table 6 shows, this feature

⁷⁵ For the stochastic solution, the equilibrium *PD* in Figure 7 is determined from the market-clearing condition (38) assuming $W_t/D_t = \rho$, to be comparable with the value this variable assumes in the vanishing risk limit. The figure assumes the parameters implied by the estimated model with diagonal matrix from Table 3.

TABLE 6—FURTHER DATA AND MODEL MOMENTS

| | Data moment | Standard deviation | Estimated model (Table 3, diagonal matrix) | | RE version estimated (Table 3, diagonal matrix) | | Estimated model with supply shocks | |
|--------------------------------------|----------------|-----------------------|--|-----------------|---|-----------------|---------------------------------------|-----------------|
| | | | Moment | <i>t</i> -ratio | Moment | <i>t</i> -ratio | Moment | <i>t</i> -ratio |
| Standard deviation risk-free rate | 1.11 | 0.24 | 0.83 | −1.16 | 0.62 | −2.04 | 0.62 | −2.05 |
| <i>Autocorrelations</i> | | | | | | | | |
| Excess stock return | 0.12 | 0.046 | 0.53 | 9.08 | 0.33 | 4.78 | 0.12 | 0.18 |
| Stock return | 0.13 | 0.045 | 0.54 | 9.21 | 0.04 | −1.87 | 0.14 | 0.37 |

arises not solely due to the presence of subjective beliefs: the rational expectations versions of the model also overpredicts the autocorrelation of excess stock returns.

The failure of the subjective belief model to deliver a near zero high-frequency autocorrelation for returns is related to the fact that the low frequency returns generated by learning also show up at high frequency because the model features few transitory shocks to stock prices. As we show below, the subjective belief model becomes consistent with the observed value of the quarterly autocorrelations of stock returns and excess stock returns, when adding small i.i.d. disturbances to the model.

To illustrate this point in a simple way, we consider in online Appendix A13 a model where equilibrium stock supply in period t satisfies

$$S_t = e^{\varepsilon_t^s},$$

where $\varepsilon_t^s \sim iin(-\sigma_{\varepsilon^s}^2/2, \sigma_{\varepsilon^s}^2)$ denotes an exogenous shock to the supply of stocks. This shock can represent extraneous demand for stocks, e.g., the one arising from noise or liquidity traders.⁷⁶ This generalized setup nests the model without noise traders for the special case $\sigma_{\varepsilon^s} = 0$.

As it turns out, small positive values for the standard deviation σ_{ε^s} bring the model fully in line with the observed autocorrelations. The last two columns in Table 6 illustrate this fact and report the model implied autocorrelations, when setting $\sigma_{\varepsilon^s} = 1.25 \cdot 10^{-3}$, which implies that the standard deviation of aggregate stock supply is 0.125 percent of the average outstanding amount of stocks (which is equal to 1). Table 6 shows that the autocorrelations are now fully consistent with those observed in the data, while the standard deviation of the risk-free rate is hardly affected. Moreover, as shown in online Appendix A13, the presence of stock supply shocks has only small effects on the other asset-pricing moments reported in Table 3.

⁷⁶ Alternatively, the shocks ε_t^s may capture changes to asset float, as discussed in Ofek and Richardson (2003) and Hong, Scheinkman, and Xiong (2006). In any case, they capture (exogenous) stock demand or supply that is not coming from the consumers described in equation (14).

B. Subjective Consumption Plans and the Sharpe Ratio

A somewhat surprising feature of the estimated models in Table 3 is that they give rise to a fairly high Sharpe ratio, despite the fact that the estimated risk aversion is relatively low. Specifically, the estimated models imply that the quarterly unconditional Sharpe ratio, $(E[r^s] - E[r^b])/\text{std}[r^s]$, is close to 0.1. Since the model implied equity premium is too low, this value falls significantly short of the value of Sharpe ratio in the data (0.22). Yet, it significantly exceeds the value it would assume if agents held rational price expectations. Under RE, the Sharpe ratio would be tiny and approximately be given by⁷⁷

$$(45) \quad \frac{E[R] - E[R^b]}{\text{std}[R]} \approx \gamma \cdot E[\text{std}_t(C_{t+1}/C_t)] = 0.00576.$$

In the presence of subjective price beliefs, however, the Sharpe ratio is approximately equal to

$$(46) \quad \frac{E[R_{t+1}] - E[R_t^b]}{\text{std}[R]} \approx \underbrace{\frac{E[R_{t+1}] - E[E_t^P[R_{t+1}]]}{\text{std}[R]}}_{\text{Subj. return pessimism}} + \underbrace{\gamma \frac{E[\text{std}_t^P[C_{t+1}/C_t]]}{E[\text{std}_t[C_{t+1}/C_t]]}}_{\substack{\text{Ratio of} \\ \text{subj. to obj.} \\ \text{consumption volatility}}} E[\text{std}_t[C_{t+1}/C_t]],$$

which illustrates the existence of two additional factors affecting the Sharpe ratio: subjective return pessimism, as present in models where agents entertain robustness concerns, e.g., Cogley and Sargent (2008), and the relative volatility of subjective consumption plans compared to the ones emerging under rational expectations. Subjective return pessimism contributes to the Sharpe ratio because pessimism depresses stock prices and thereby generates a higher equity premium ex post. Subjective consumption volatility affects the equity premium for standard reasons.

For the estimated models from Table 3, we find that about one-third of the model's Sharpe ratio is due to subjective return pessimism and about two-thirds due to the second term on the RHS of equation (46).⁷⁸ In both estimated models, the second component is so large because the average subjective standard deviation of consumption growth is about 12.5 times larger than the average objective standard deviation.⁷⁹

⁷⁷ See online Appendix A14 for a derivation of equations (45) and (46).

⁷⁸ The approximate Sharpe ratio decomposition (46) works quantitatively well for the estimated models from Table 3: the actual Sharpe ratios are 10.7 percent and 10.2 percent for the model estimated using a diagonal and an efficient matrix, respectively, while the terms on the RHS in equation (46) sum to 10.5 percent and 10.9 percent.

⁷⁹ The precise ratios are 12.87 (12.64) for the model from Table 3 estimated using a diagonal (efficient) matrix. For both models, we have $E[\text{std}_t[C_{t+1}/C_t]] = 0.00273$.

Analyzing which future shocks contribute in agents' mind to this high level of subjectively expected consumption volatility, we find that transitory price growth shocks are the main culprit, i.e., the perceived shocks ε_{t+1} in the subjective price evolution equation (26). These shocks move prices permanently up (or down), without an associated change in fundamentals, and prompt agents to make contingent plans to sell (or buy) stocks in the future in response to these shocks. These expectations of future trade increase subjective expected consumption volatility, while the representative agent assumption insures that agents will never actually buy or sell stocks in equilibrium. As a result, subjective and objective consumption volatility diverge considerably.

The reason for this discrepancy is largely due to the fact that the homogeneous agent model imposes zero trade in equilibrium. Equilibrium consumption is then not affected by stock prices. As we now illustrate, one can resolve this discrepancy by introducing heterogeneity among agents.

We show this using two simple heterogeneous agent extensions of our baseline model. The first extension studies an ad hoc modification in which agents' beliefs at time t are given by

$$(47) \quad \ln m_t^i = \ln m_t + \ln \mu_t^i,$$

where $\ln \mu_t^i \sim iiN(0, \sigma_\mu^2)$ is an idiosyncratic and *transitory* shock to agent i 's capital gain optimism and where m_t evolves as in the baseline model. The second extension considers a fully microfounded extension, where—in line with the belief updating equation (39)—agents' expectations evolve according to

$$(48) \quad \ln m_t^i = \ln m_{t-1}^i + g(\ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1}^i) + g \ln \varepsilon_t^{1,i},$$

where $(g \ln \varepsilon_t^{1,i}) \sim iiN(0, \sigma_{\varepsilon,1}^2)$ is idiosyncratic with rather *persistent* effects on beliefs.⁸⁰ In both extensions, belief heterogeneity induces trade in equilibrium, which increases the objective volatility of individual consumption growth relative to the representative agent setting. We then calibrate the variances σ_μ^2 and $\sigma_{\varepsilon,1}^2$ such that the objective and subjective standard deviations are approximately aligned ($E[\text{std}_t^P[C_{t+1}^i/C_t^i]] \approx E[\text{std}_t[C_{t+1}^i/C_t^i]]$).

Table 7 reports the asset-pricing moments and t -ratios for both heterogeneous agent extensions.⁸¹ As becomes clear from Table 7, the extended models can replicate the asset-pricing moments equally well as the baseline model, but also align the subjective and objective standard deviations of consumption growth. The ability of the model to generate a low risk-free rate is actually improved relative to the representative agent model, with the average risk-free rate even becoming slightly negative for the model with persistent shocks.⁸² Overall, Table 7 illustrates the

⁸⁰ As is clear from online Appendix A7, the shock $\ln \varepsilon_t^{1,i}$ represents information that the agent receives about the perceived transitory price shock. Since these shocks exist only in subjective terms, the baseline model sets $\ln \varepsilon_t^i = \ln \varepsilon_t = 0$ for all t .

⁸¹ The table uses the model parameterization of the estimated model from Table 3 (diagonal matrix). For the case with persistent belief shocks, we slightly adjust the updating gain g to improve the match with the asset-pricing moments. The simulations use 51 different agents.

⁸² The strong effect on the risk-free rate is due to the tight borrowing limits ($B_t^i \geq 0$) that we impose: see the discussion in Section III. The effect would likely be less strong for less strict borrowing limits, but this would require fundamental changes to the solution approach, including an increase in the number of state variables.

TABLE 7—HETEROGENEOUS AGENT MODEL WITH EQUILIBRIUM TRADE

| | US data 1946:I–2012:I (quart. real values) | Subjective belief model (transitory belief shocks) | | Subjective belief model (persistent belief shocks) | |
|---------------------------------------|--|---|-----------------|--|-----------------|
| | Moment | Moment | <i>t</i> -ratio | Moment | <i>t</i> -ratio |
| $E[PD]$ | 139.8 | 113.5 | −1.05 | 120.6 | −0.77 |
| $\text{std}[PD]$ | 65.2 | 85.9 | 1.40 | 90.8 | 1.73 |
| $\text{corr}[PD_t, PD_{t-1}]$ | 0.98 | 0.98 | 0.63 | 0.98 | 0.51 |
| $\text{std}[R]$ | 8.00 | 7.37 | −1.58 | 7.93 | −0.16 |
| c | −0.0041 | −0.0047 | −0.53 | −0.0064 | −1.81 |
| R^2 | 0.25 | 0.18 | −0.47 | 0.25 | 0.05 |
| $E[R] - 1$ | 1.89 | 1.80 | −0.19 | 1.87 | −0.06 |
| $E[R^b] - 1$ | 0.13 | 0.34 | 1.26 | −0.65 | −4.58 |
| <i>UBS survey data</i> | | | | | |
| $\text{corr}[PD_t, E_t^P R_{t,t+4}]$ | 0.79 | 0.79 | −0.08 | 0.85 | 0.78 |
| <i>Consumptive volatility</i> | | | | | |
| $E[\text{std}_t^P C_{t+1}^i / C_t^i]$ | | 3.38 | | 4.55 | |
| $E[\text{std}_t C_{t+1}^i / C_t^i]$ | | 3.23 | | 4.99 | |
| <i>Estimates</i> | | | | | |
| g | | 0.0282 | | 0.0282 | |
| δ | | 0.99514 | | 0.99514 | |
| γ | | 2.03 | | 2.03 | |
| σ_μ | | 0.0030 | | — | |
| $\sigma_{\varepsilon,1}$ | | — | | 0.0018 | |

Notes: The table reports US asset-pricing moments (column 2) using the data sources described in online Appendix A1, the moments and *t*-ratios of the subjective belief model with idiosyncratic transitory belief shocks (columns 3 and 4, see equation (47)), and the moments and *t*-ratios of the subjective belief model with idiosyncratic persistent belief shocks (columns 5 and 6, see equation (48)). $E[\text{std}_t^P C_{t+1}^i / C_t^i]$ denotes the unconditional expectation of the subjective standard deviation of consumption growth and $E[\text{std}_t C_{t+1}^i / C_t^i]$ the unconditional expectation of the objective standard deviation of consumption growth. The standard deviation of the idiosyncratic belief shocks σ_μ and σ_ε has been chosen so as to approximately equate subjective and objective standard deviation of consumption growth. See Table 3 for a description of the remaining labels used in the first column.

robustness of our quantitative results toward allowing for investor heterogeneity. Adam et al. (2015) explore in greater detail a heterogeneous agent version of the present model and show that the model successfully matches important stylized facts regarding trading volume.

C. Expectational Errors

Since our model matches stock price behavior and the comovement pattern between survey expectations and stock prices, agents' expectations in the model fail to satisfy the rational expectations hypothesis, in line with what has been documented for the survey evidence. To document the quantitative importance of this fact, this section applies the RE tests from Section IIB to the subjectively expected one-quarter-ahead capital gain expectations from our estimated models. In particular, we test the acceptance rates of the rational expectations hypothesis ($H_0: c = \mathbf{c}$) at various significance intervals and various sample lengths, using the test statistic

TABLE 8—REJECTION FREQUENCIES FOR THE RE HYPOTHESIS ($H_0: c = c$)
FOR THE ESTIMATED ASSET-PRICING MODELS

| | Significance level | Sample length in quarters | | | | | |
|--|--------------------|---------------------------|-------|-------|-------|-------|-------|
| | | 25 | 50 | 75 | 100 | 150 | 200 |
| Model from Table 7 (diagonal matrix) | 1% | 0.828 | 0.944 | 0.971 | 0.981 | 0.984 | 0.987 |
| | 5% | 0.900 | 0.970 | 0.981 | 0.987 | 0.988 | 0.990 |
| Model from Table 7 (efficient matrix) | 1% | 0.779 | 0.938 | 0.968 | 0.975 | 0.975 | 0.989 |
| | 5% | 0.892 | 0.974 | 0.986 | 0.981 | 0.980 | 0.991 |

Note: The reported rejection frequencies are based on the rational expectations test developed in Section IIB of the paper and include the small sample bias correction stated in Proposition 1.

described in Proposition 1.⁸³ Table 8 reports the outcomes.⁸⁴ It shows that at conventional significance levels the rational expectations hypothesis can be rejected with high likelihood, even for relatively short sample lengths. This is in line with the empirical evidence documented in Section II.

IX. Historical PD Ratio and Survey Evidence

This section shows that the estimated subjective belief model successfully replicates the low-frequency movements of the postwar US *PD* ratio, as well as the available time series of survey expectations. We illustrate this point using the model from Table 5, which has been estimated under the restriction that the wage-dividend ratio displays close to unit root behavior ($p = 0.999$). This is motivated by the fact that estimated models that imply more mean reversion in the wage-dividend ratio, say the ones from Table 3, cannot capture the time series behavior of the *PD* ratio after the year 1990. After 1990, the wage-dividend ratio started to persistently diverge from a value close to its sample mean of 22 to a value close to 12 at the end of the sample period in 2012.

To determine the model fit, we feed the historically observed price growth observations into the model's belief updating equation (39), so as to obtain a model-implied belief process.⁸⁵ We then combine this belief series with the historically observed wage-dividend ratio to obtain from the model's equilibrium pricing function the model implied prediction for the postwar *PD* series. Formally, this is done by plugging implied expectations and wage-dividend ratio in equation (38) and backing out the corresponding *PD* ratio. In doing so, we choose the unobserved price growth belief at the start of the sample period ($m_{1946:1}$) so as to minimize the sum of absolute deviations of the model-implied *PD* ratio from the *PD* ratio in the data.⁸⁶

⁸³ We include the small sample bias correction reported in Proposition 1.

⁸⁴ Rejection frequencies have been computed from 1,000 random samples of the specified sample length that were randomly drawn from a time series of 100,000 simulated model periods, with the first 10,000 periods being discarded as burn-in.

⁸⁵ Since we cannot observe the shocks $\ln \varepsilon_t^1$ in (39), we set them equal to 0. Gaps between the model predicted and actual *PD* ratio may thus partly be due to these shocks.

⁸⁶ This delivers initial price growth expectations in 1946:I of -1.7 percent.



FIGURE 8. PRICE GROWTH EXPECTATIONS IMPLIED BY BAYESIAN UPDATING
AND HISTORICAL STOCK PRICE GROWTH
(Estimated Model from Table 5, $p = 0.999$), 1946:1–2012:1

The resulting belief sequence is depicted in Figure 8. Figure 9 graphs the model implied PD ratio together with the PD ratio from the data. It reveals that the model captures a lot of the low-frequency variation in the historically observed PD ratio. It captures particularly well the variations before the year 2000, including the strong run-up in the PD ratio from the mid 1990s to the year 2000. The model also predicts a strong decline of the PD ratio after the year 2000, but overpredicts the decline relative to the data.

The gap after the year 2000 emerging in Figure 9 is hardly surprising, given the empirical evidence presented in Figure 4, which shows that the relationship between the PD ratio and the expectations implied by equation (13) has shifted upward in the data following the year 2000. While we can only speculate about potential reasons causing this shift, the exceptionally low real interest rates implemented by the Federal Reserve following the reversal of the tech stock boom and following the collapse of the subsequent housing boom may partly contribute to the observed discrepancy. This suggests that incorporating the asset-pricing effects of monetary policy decisions might improve the model fit. This is, however, beyond the scope of the present paper.

Figure 10 depicts the model-implied price growth expectations and the price growth expectations from the UBS survey.⁸⁷ While the model fits the survey data rather well, it predicts after the year 2003 considerably lower capital gains expectations, which partly explains why the model underpredicts the PD ratio in Figure 9 toward the end of the sample period. While the expectations gap in Figure 10 narrows considerably after the year 2004, this fails to be the case for the PD ratio in

⁸⁷ See footnote 35 for how to compute price growth expectations from the UBS survey.

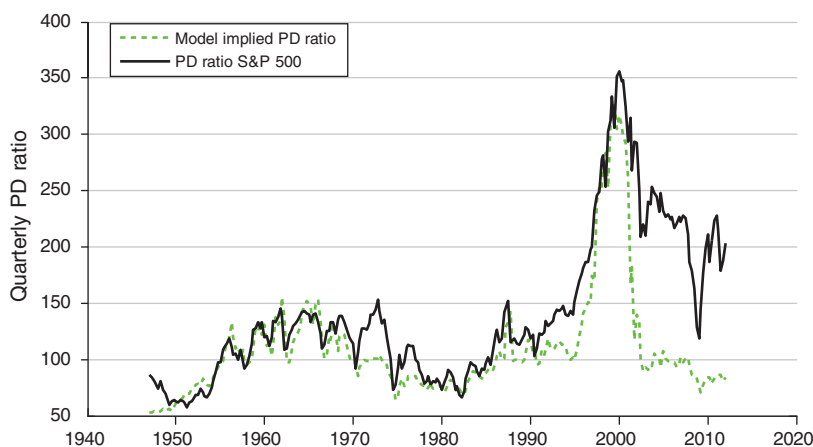


FIGURE 9. HISTORICAL PD RATIO: DATA VERSUS MODEL, 1946:I–2012:I
(Estimated Model from Table 5, $p = 0.999$)

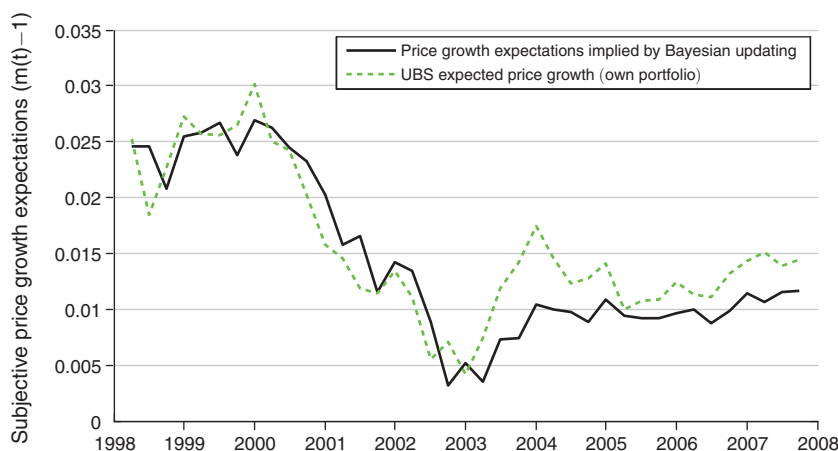
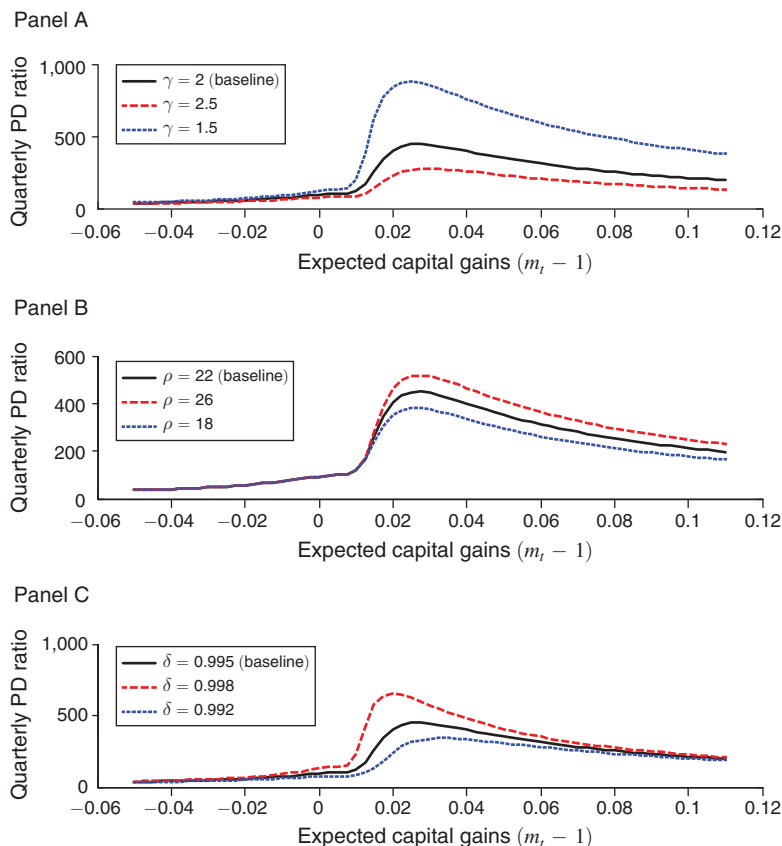


FIGURE 10. PRICE GROWTH EXPECTATIONS: UBS SURVEY VERSUS BAYESIAN UPDATING MODEL
(Estimated Model from Table 5, $p = 0.999$), 1998:II–2007:IV

Figure 9. Underprediction of expected price growth thus explains only partly the deterioration of the fit of the PD ratio toward the end of the sample period.

X. Robustness Analysis

This section explores the robustness of our main findings to changing key model parameters and to using more general price belief systems. The next section studies the pricing effects of alternative model parameters. Section XB discusses the implications of more general price belief systems that incorporate mean-reversion of the PD ratio. Finally, Section XC considers the pricing and welfare implications of introducing agents who use current price information for updating beliefs.

FIGURE 11. EFFECTS OF (γ, ρ, δ) ON THE EQUILIBRIUM PD FUNCTION

A. Model Parameterization

This section studies how the model's ability to generate boom and bust dynamics in stock prices depends on key model parameters. The equilibrium pricing function for our baseline parameterization, depicted in Figure 5, allows for self-reinforcing stock price boom and bust dynamics because the price dividend ratio increases initially strongly with capital gain optimism. This section explores the robustness of this feature by studying how the equilibrium pricing is affected by the coefficient of relative risk aversion (γ), the discount factor (δ), and the average wage to dividend income ratio (ρ).

Figure 11 depicts the equilibrium pricing function for alternative parameter choices. Each panel plots the pricing function from our baseline parameterization, as well as those generated by increasing or decreasing the values for γ , δ , and ρ . Panel A, for example, shows that lowering (increasing) the coefficient of relative risk aversion increases (reduces) the hump in the PD function and moves it to the left (right), thereby causing asset price booms to become more (less) likely and larger in size. Similar effects are associated with increasing (decreasing) the wage to dividend income ratio (ρ) and with increasing (decreasing) the discount factor (δ).

Overall, Figure 11 shows that the model can produce hump-shaped equilibrium *PD* functions over a fairly wide set of parameter specifications.

B. Generalized Belief System

This section considers a generalized price belief system, which implies that investors expect the *PD* ratio to eventually mean-revert over time. This is motivated by the fact that the price belief system (26)–(27) employed in the main part of the paper, along with knowledge of the dividend process, does not imply that agents expect the log of the *PD* ratio to mean revert over time.

We find that the pricing implications of the model do not depend on the expected long-run behavior for the *PD* ratio and that very similar equilibrium pricing functions can be generated by belief systems that imply persistent changes in the *PD* ratio but where the *PD* ratio is ultimately mean reverting.

It is important to note that the survey data provide little support for mean-reverting price growth expectations over the available forecast horizons and thus for mean reversion in the expected *PD* ratio. In particular, the last four rows in Table 1A show that the regression coefficient \hat{c} obtained from regressing the expected 10 year ahead capital gain forecasts from the Shiller survey on the *PD* ratio is more than 10 times larger than the corresponding regression coefficient obtained from regressing the one year ahead forecast on the *PD* ratio. The expected annualized ten year price growth thus reacts stronger to movements in the *PD* ratio than the expected one year price growth. This shows that one should incorporate only a mild degree of mean reversion into subjective price growth beliefs, as the belief system would otherwise become inconsistent with the survey evidence.

To analyze the effects of mean reversion in price beliefs, we now assume that investors perceive prices to evolve according to

$$(49) \quad \ln P_{t+1} = \ln \beta_{t+1} + \ln P_t + (1 - \eta_{PD})(\ln PD - \ln P_t / D_t) + \ln \varepsilon_{t+1},$$

$$(50) \quad \ln \beta_{t+1} = (1 - \eta_\beta) \ln \beta^D + \eta_\beta \ln \beta_t + \ln \nu_{t+1},$$

where $\ln PD$ denotes the perceived long-run mean of the log *PD* ratio and $\eta_{PD} \in [0, 1]$, $\eta_\beta \in [0, 1]$ are given parameters. For $\eta_{PD} = \eta_\beta = 1$ these equations deliver the benchmark price belief system (26)–(27) studied in the main part of the paper.⁸⁸ For $\eta_\beta < 1$, equation (50) implies that agents expect mean reversion in the persistent price growth component $\ln \beta_t$ toward the mean growth rate of dividends ($\ln \beta^D$); if in addition $\eta_{PD} < 1$, equation (49) implies that agents expect $\ln P_t / D_t$ to eventually return to its long-run mean $\ln PD$.⁸⁹

Suppose that $\eta_\beta < 1$ and $\eta_{PD} < 1$, that agents are optimistic about future capital gains, i.e., they believe β_t to be above β^D , and that agents observe a *PD* ratio above its long-run mean ($P_t / D_t > PD$). Provided η_β and η_{PD} are sufficiently close to 1, equations (49)–(50) imply that agents expect a fairly persistent boom in the *PD* ratio, as is the case with the benchmark belief system (26)–(27). Yet, unlike with the

⁸⁸ As before, we assume that agents have rational expectations about the dividend and wage income processes.

⁸⁹ This can be seen by subtracting (15) from (49).

benchmark, they also expect a price bust further down the road, because the PD ratio is expected to eventually return to its long-run value.

Equations (49) and (50) jointly imply that optimal belief updating about the unobserved persistent stock price growth component $\ln \beta_t$ is described by a generalized version of equation (39), which states that the posterior mean $\ln m_t \equiv E^P(\ln \beta_t | P^t)$ in steady state evolves according to⁹⁰

$$(51) \quad \ln m_t = (1 - \eta_\beta) \ln \beta^D + \eta_\beta \ln m_{t-1} + g \left(\frac{\ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1}}{-(1 - \eta_{PD})(\ln PD - \ln P_{t-2} / D_{t-2})} \right) + g \ln \varepsilon_t^1.$$

Using the generalized belief and updating equations, Figure 12 depicts the impact of different perceived values for η_{PD} and η_β on the equilibrium pricing function when setting the perceived long-run mean $\ln PD$ in (49) equal to the perfect foresight value of the PD ratio.⁹¹ Panel A plots the effects of decreasing η_{PD} below one, while keeping $\eta_\beta = 0.9999$. For the considered values of η_{PD} , agents expect the log PD ratio to mean revert by 1 percent, 2 percent, or 3 percent per year toward its long-run mean ($\ln PD$). The figure also plots the outcome when there is virtually no mean reversion ($\eta_{PD} = 0.9999$). Panel A shows that by introducing mean reversion, one pushes the peak of the equilibrium PD ratio to the right and also lowers its height. The shift to the right occurs because agents only expect a persistent boom in the PD ratio if the increase in the PD ratio implied by the persistent growth component ($\ln \beta_t$) outweighs the mean reversion generated by the negative feedback from the deviation of the PD from its long-run value in equation (49). The downward shift in the PD ratio occurs because mean reversion causes investors to expect lower and eventually negative returns sooner.

Panel B in Figure 12 depicts the effects of decreasing η_β below 1, while keeping $\eta_{PD} = 0.9999$. As before, we consider values for η_β that imply virtually no mean reversion and mean reversion by 1 percent, 2 percent, and 3 percent per year toward the long-run value ($\ln \beta^D$). The panel shows that the pricing implications are very similar to those of decreasing η_{PD} .

Finally, panel C depicts the effects of jointly decreasing η_β and η_{PD} . The pricing implication of such simultaneous changes turn out to be considerably stronger, compared to the case where only one of the persistence parameters decreases. Figure 13 illustrates why this sharp difference occurs. It graphs the expected path of the PD ratio for different parameter combinations (η_{PD}, η_β) .⁹² If either η_{PD} or η_β are close to 1, agents expect a rather prolonged stock price boom that is expected to revert only in the distant future. Expected stock returns are thus high for many periods

⁹⁰Since equation (49) only introduces an additional observable variable into (26) and equation (50) only adds a known constant and known mean reversion coefficient relative to (27), the arguments delivering equation (39) as optimal Bayesian updating directly generalize to equation (51).

⁹¹We assume that σ_ε^2 and σ_ν^2 assume the same values as in the baseline specification, so that the gain parameter also remains unchanged at $g = 0.02515$. This parameterization makes sure that—for the values of η_β and η_{PD} considered below—the perceived standard deviation for stock price growth implied by equations (50)–(49) approximately matches the standard deviation of stock price growth in the data.

⁹²The figure assumes that the equilibrium PD ratio initially equals $PD_0 = 150$, i.e., it is above its long-run value, and that agents are mildly optimistic about future capital gains with $\ln m_0 = 1\% > \ln \beta^D$.

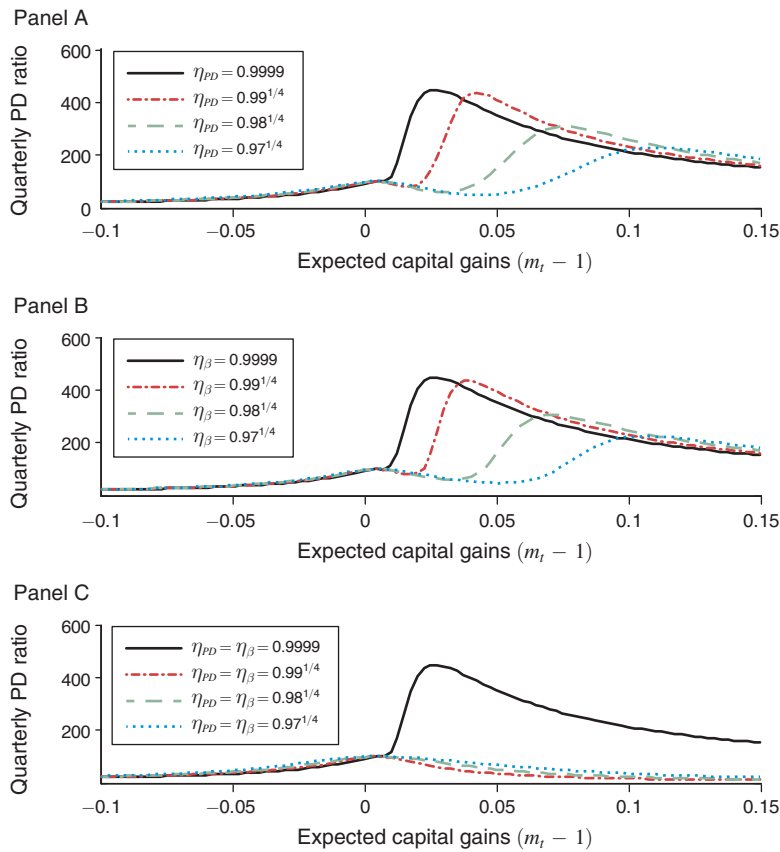


FIGURE 12. EQUILIBRIUM PRICING FUNCTION (*Generalized Belief System*)

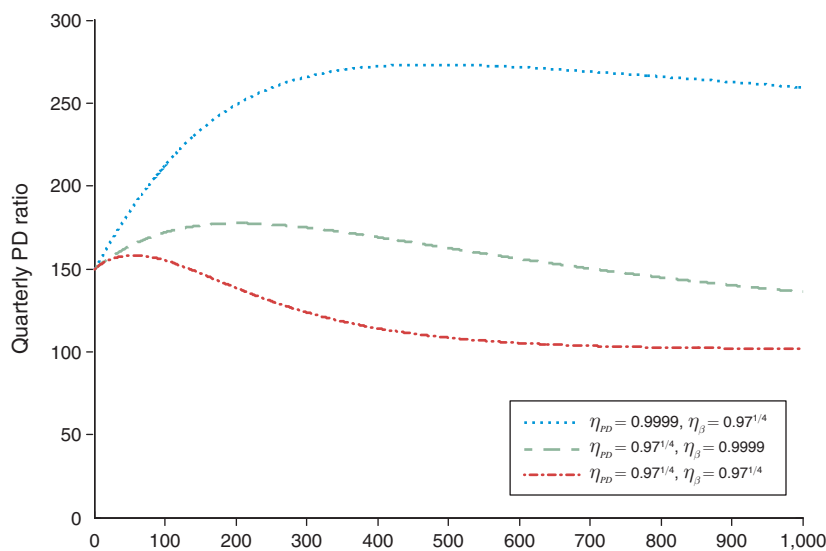


FIGURE 13. PERSISTENCE PARAMETERS AND EXPECTED PD PATH (*Generalized Belief System*)

before they turn negative. This differs notably from the case where both persistence parameters fall significantly below one ($\eta_{PD} = \eta_{\beta} = 0.97^{\frac{1}{4}}$). The expected stock price boom is then much smaller and considerably more short-lived, so that returns are lower and expected to become negative earlier. From the discussion following equation (42), it should be clear that the implied path for expected returns then cannot sustain a high *PD* ratio as an equilibrium outcome.

Summing up, the model continues to give rise to hump-shaped equilibrium pricing functions, even if agents ultimately expect mean reversion in the *PD* ratio, provided the generalized belief system implies that agents expect elevated capital gains to be sufficiently persistent.

C. Current Price Information for Belief Updating

The baseline model specification postulates a belief structure that makes it subjectively optimal for agents to update beliefs m_t based on price information up to period $t - 1$ only, see equation (39). The present section demonstrates that lagged belief updating is also objectively optimal for agents: belief updating with current price observations robustly generates lower experienced utility for agents.⁹³ In this sense, the baseline specification with lagged belief updating can be interpreted as the limiting outcome of a setting in which agents choose—based on experienced utility—whether to use current prices for updating beliefs.

To show that lagged updating generates objectively higher utility, we consider an extended model setup where a share $\alpha \in [0, 1]$ of investors updates beliefs using current price growth information, with the remaining share $1 - \alpha$ using lagged price growth information.⁹⁴ While the beliefs of lagged updaters are predetermined, the beliefs of current updaters vary simultaneously with the stock price P_t . The latter creates a potential for multiple market-clearing equilibrium price and belief pairs.

To assess the implications of simultaneous belief updating, we consider the estimated benchmark parameters from Table 3 (diagonal matrix) and different values for $\alpha \in [0, 1]$. To select between equilibrium prices in periods where multiple equilibrium price exist, we consider two alternative selection rules.⁹⁵ The first rule selects the equilibrium price that is closest to the previous period's market-clearing price. The second selection rule selects the price that is furthest away from the previous market-clearing price.

Table 9 reports the expected discounted utility of agents that use current and lagged belief updating for different values of α and for the two considered selection rules.⁹⁶ It shows that independently of the selection rule and independently of the share of current updaters α , utility of lagged updaters always exceeds that of current

⁹³ This finding is in line with results reported in Adam et al. (2015), who show that agents whose beliefs are more reactive to price growth observations tend to do worse than agents whose beliefs display less sensitivity.

⁹⁴ Updaters using current price information update beliefs according to equation (39), but replace $\ln P_{t-1} / \ln P_{t-2}$ by $\ln P_t / \ln P_{t-1}$ on the right-hand side.

⁹⁵ In the vast majority of cases, we find three market-clearing equilibrium prices, conditional on there being multiplicity. In less than 0.1 percent of the periods with multiplicities we find five market-clearing prices.

⁹⁶ The table reports the unconditional expectation of discounted consumption utility using the objective distribution for consumption, as realized in equilibrium. Online Appendix A15 reports the associated asset-pricing moments.

TABLE 9—EXPECTED UTILITY OF CURRENT AND LAGGED BELIEF UPDATERS

| Share of current updaters (α) | Closest price | | Furthest price | |
|---|--------------------|---------------------|--------------------|---------------------|
| | Lagged updaters | Current updaters | Lagged updaters | Current updaters |
| 0.0 | −4.02 | −4.07 | −4.02 | −4.07 |
| 0.1 | −4.02 | −4.06 | −4.02 | −4.06 |
| 0.2 | −4.01 | −4.06 | −4.01 | −4.06 |
| 0.3 | −4.01 | −4.05 | −4.01 | −4.06 |
| 0.4 | −4.01 | −4.05 | −4.00 | −4.06 |
| 0.5 | −4.00 | −4.05 | −3.99 | −4.06 |
| 0.6 | −4.00 | −4.04 | −3.98 | −4.06 |
| 0.7 | −3.99 | −4.04 | −3.97 | −4.06 |
| 0.8 | −3.99 | −4.03 | −3.89 | −4.11 |
| 0.9 | −3.97 | −4.03 | −3.39 | −4.24 |
| 1.0 | −3.95 | −4.02 | −2.46 | −4.02 |

Notes: The table reports the objectively expected utility for agents that use current and one period lagged prices to update beliefs for different equilibrium selection rules, as described in Section XC, and different shares α of current updaters. Since $u(C) < 0$, expected utility is always negative. The model is parameterized using the estimated parameters from Table 3 (diagonal matrix).

updaters. Current updaters thus have an incentive to switch to lagged updating, i.e., to the setting considered in our baseline specification.

XI. Conclusions

We present a model with rationally investing agents that gives rise to market failures in the sense that the equilibrium stock price deviates from its fundamental value. These deviations take the form of asset price boom and bust cycles that are fueled by the belief-updating dynamics of investors who behave optimally given their imperfect knowledge about the behavior of stock prices. Optimal belief updating also causes investors' subjective capital gain expectations to comove positively with the price-dividend ratio, consistent with the evidence available from investor surveys.

As we argue, these features cannot be replicated within asset-pricing models that impose rational price expectations. Moreover, the developed statistical tests show that the behavior of survey return expectations is incompatible with the rational expectations hypothesis.

Taken together, this suggests that asset price dynamics are to a large extent influenced by investors' subjective optimism and pessimism, i.e., the asset price fluctuations observed in the data are to a large extent inefficient. The inefficiency arises because equilibrium stock prices are determined by the sum of dividends that is discounted using the stochastic discount factor implied by investors' subjective consumption plans, which are influenced by investors' subjective price beliefs. This differs from the standard setup under RE, where stock prices equal the sum of dividends discounted by the objective stochastic discount factor.

Due to the simplicity of the setup, these inefficient price fluctuations do not yield adverse welfare implications in our baseline model.⁹⁷ For models incorporating investor heterogeneity, e.g., the extension presented in Section VIIIB, or models

⁹⁷This is true if one evaluates welfare using ex post realized consumption.

featuring endogenous output or stock supply processes, stock price fluctuations may have significant effects on welfare. Exploring these within a setting that generates quantitatively credible amounts of asset price fluctuations appears to be an interesting avenue for further research. Such research will in turn lead to further important questions, e.g., whether policy can and should intervene with the objective to stabilize asset prices.

In deriving our results, we assumed that all agents in the economy become more (or less) optimistic when observing capital gains above (or below) their expectations. While the quantitative model predictions survive when investors are heterogeneous in the degree to which they respond to observed capital gains (see Adam et al. 2015), it appears of interest to assess the potential price impact generated by speculators with rational price expectations. While rational speculators can contribute to price destabilization, as in De Long et al. (1990), they may also help with price stabilization, as in Barberis et al. (2015). Exploring this issue further, especially in connection with the limits to arbitrage emphasized in Shleifer and Vishny (1997), appears to be a fruitful avenue for further research.

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