# Online Appendix to

# The Optimal Inflation Target and the Natural Rate of Interest [1]

Philippe Andrade

Federal Reserve Bank of Boston

Jordi Galí

CREI, UPF and Barcelona GSE

Hervé Le Bihan

Banque de France

Julien Matheron

Banque de France

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## A Various long-run and optimal inflation rates considered

Table A.1: Various notions of long-run and optimal inflation in the model

| | |
|---|---|
| $\pi$ | Any inflation target, used to define the "inflation gap" that enters the Taylor rule |
| $\mathbb{E}(\pi_t)$ | Average realized inflation, might differ from $\pi$ due to ZLB |
| $\pi^\star(\theta)$ | Inflation target that minimizes the loss function given a structural parameters $\theta$ |
| $\pi^\star(\bar{\theta})$ | $\pi^\star$ assuming parameters at post. mean |
| $\pi^\star(median(\theta))$ | $\pi^\star$ assuming parameters at post. median |
| $\bar{\pi}^\star$ | average of $\pi^\star(\theta)$ over the posterior distribution of $\theta$, i.e., $\int_\theta \pi^\star(\theta)p(\theta\|X_T)\mathrm{d}\theta$ |
| Median($\pi^\star$) | Median of $\pi^\star(\theta)$ over the posterior distribution |
| $\pi^{\star\star}$ | Inflation target that minimizes the average loss function over the posterior distribution of $\theta$ |

[1] Andrade: philippe.andrade@bos.frb.org. Galí: jgali@crei.cat. Le Bihan: herve.lebihan@banque-france.fr. Matheron: julien.matheron@banque-france.fr. The views expressed herein are those of the authors and do not necessarily reflect those of the Banque de France, the Eurosystem, the Federal Reserve Bank of Boston or the Federal Reserve System.

# B Illustrating model properties: moments, IRF to monetary policy shock

This section illustrate basic properties of the estimated baseline model.

Table B.1: Moments of key variables

**Data 1985Q2-2008Q3**

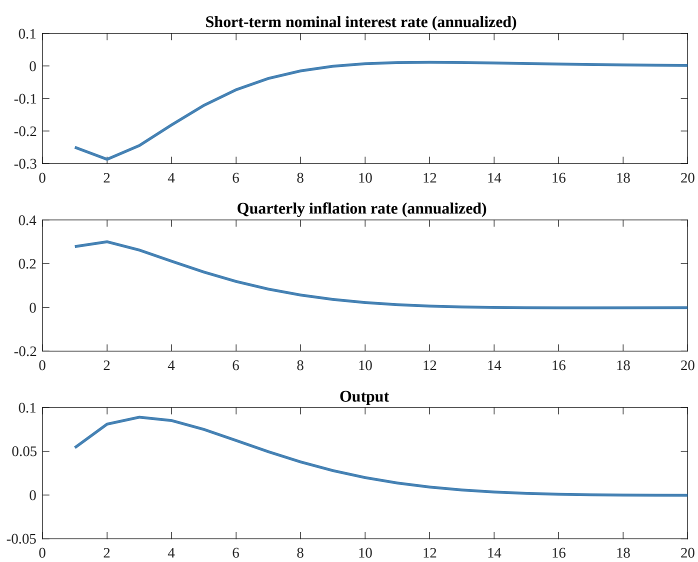
| Variable | Inflation | 4-Quarter -Inflation | Output gap | Output growth | Interest rate |
|---|---|---|---|---|---|
| Std. dev. | 0.22 | 0.73 | — | 0.54 | 2.20 |

**Simulated Model (with ZLB constraint)**

| Variable | Inflation | 4-Quarter -Inflation | Output gap | Output growth | Interest rate |
|---|---|---|---|---|---|
| Std. dev. | 0.43 | 1.53 | 0.58 | 0.99 | 2.15 |
| | (0.11) | (0.41) | (0.14) | (0.14) | (0.14) |

**Note:** In percent. Inflation is quarterly inflation (not annualized). Interest rate is annualized. 4-Quarter inflation is the year-on-year growth rate of the price index. The model moments are based on 1000 simulations at the posterior mean. At each simulation, shocks are drawn (with replacement) from the historical shocks. The figures in parentheses are the standard deviation across bootstrap simulations.

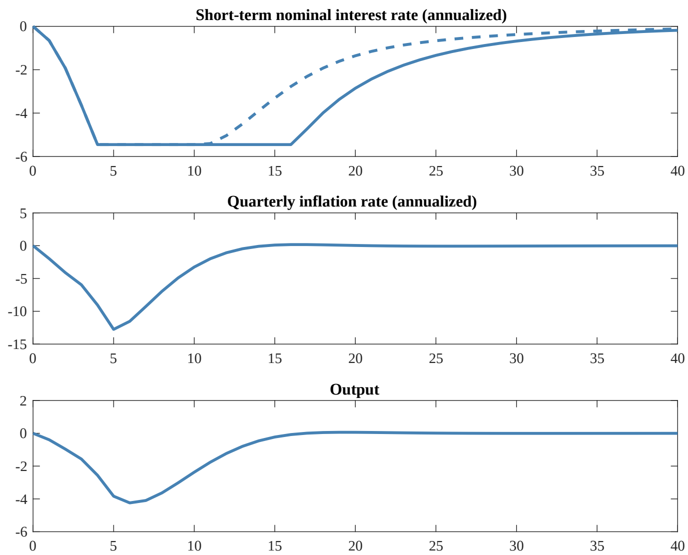
Figure B.1: Response to a monetary policy shock




**Note:** Plain line : response to a monetary policy shoxk leading to -25 basis point cut in the nominal interest on impact. Inflation is the annualized quarterly growth rate of the price index. Interest rate is annualized.

## C Illustrating the "lower for longer" property of the model policy rule

Figure C.1: Interest rate, inflation and output path in a recession with ELB scenario




**Note:** Plain line : actual model policy rule. Dashed line: illustrative interest rule featuring actual rate lagged term rather than lagged "notional rate" term. The latter rule has no feedback on the model.

In this section, we illustrate how the "lower for longer" property of the model policy rule works in practice. To this end, we assume that the model starts in steady state and is hit by a series of unexpected risk-premium shocks that drive the economy to the ZLB. Given the implied path for inflation $\hat{\pi}_t$, the output gap $\hat{x}_t$, and the notional rate $\hat{\imath}_t^n$, we reconstruct the path of an alternative interest rate $\tilde{\imath}_t$ that would obey

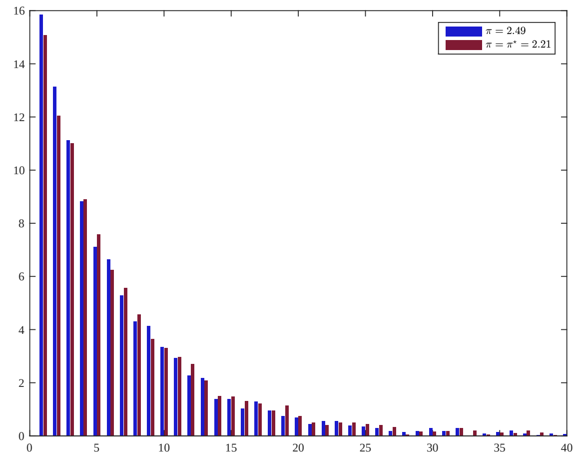
$$\tilde{\imath}_t^n = \rho_i \tilde{\imath}_{t-1} + (1-\rho_i)(a_\pi \hat{\pi}_t + a_y \hat{x}_t) + \zeta_{R,t}$$

$$\tilde{\imath}_t = \max\{\tilde{\imath}_t^n, -(\mu_z + \rho + \pi)\}.$$

In this alternative specification, the notional rate does not depend on its lagged value but rather on the lagged value of the nominal interest rate. Away from the ZLB, this has no discernible effect. However, when the economy hits the ZLB, $\tilde{\imath}_t^n$ will mechanically increase sooner than $\hat{\imath}_t^n$. Figure C.1 reports the outcome of this simulation. The solid blue line shows the paht of $\hat{\imath}_t$ while the dashed line shows the implied path for $\tilde{\imath}_t$.

# D The distribution of ZLB spells duration

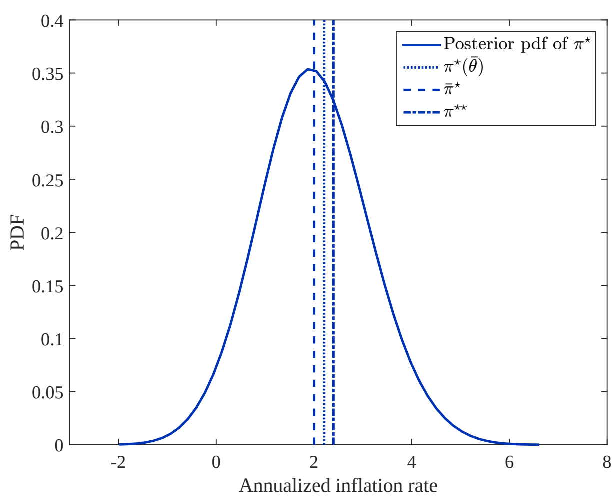
Figure D.1: Distribution of ZLB spells duration at the posterior mean




**Note:** Histograms are based on a simulated sample of 500,000 quarters. Simulations are carried out assuming in turn that the inflation target is the estimated inflation target ; and then that the inflation target is the optimal inflation target obtained using the mean of the posterior density of estimated parameters

# E The distribution of optimal inflation targets

Figure E.1: Posterior Distribution of $\pi^{\star}$




**Note:** Plain curve: PDF of $\pi^{\star}$; Dashed vertical line : Average value of $\pi^{\star}$ over posterior distribution; Dotted vertical line : Optimal inflation at the posterior mean of $\theta$; Dashed-dotted vertical line : Bayesian-theoretic optimal inflation

# F The welfare cost of inflation

Following a standard approach when assessing alternative policies, we complement our characterization of optimal inflation by providing measures of consumption-equivalent welfare gains/losses of choosing a suboptimal inflation target.

Let $\mathscr{W}(\pi)$ denote welfare under the inflation target $\pi$. It is defined as

$$\mathscr{W}(\pi) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \mathrm{e}^{\zeta_{g,t}} \log(\hat{C}_t(\pi) - \eta \hat{C}_{t-1}(\pi) \mathrm{e}^{-\zeta_{z,t}}) - \frac{\chi}{1+\nu} \int_0^1 N_t(\pi, h)^{1+\nu} dh \right] + \Psi_0(\mu_z, \zeta_z).$$

Importantly, the welfare function is stated in terms of detrended consumption. The term $\Psi_0$ captures the part of welfare that depends exclusively on $\mu_z$ and $\zeta_{z,t}$ and is not affected by changes in the inflation target.

Let us now consider a deterministic economy in which labor supply is held constant at the undistorted steady-state level $N_n$ and in which agents consume the constant level of detrended consumption $\hat{C}(\pi)$. We seek to find the $\hat{C}(\pi)$ such that this deterministic economy enjoys the same level of welfare as above. Thus

$$\mathscr{W}(\pi) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \log\left((1-\eta)\hat{C}(\pi)\right) - \frac{\chi}{1+\nu} N_n^{1+\nu} \right] + \Psi_0(\mu_z, 0).$$

Direct manipulations thus yield

$$\mathscr{W}(\pi) = \frac{1}{1-\beta} \left[ \log\left((1-\eta)\hat{C}(\pi)\right) - \frac{\chi}{1+\nu} N_n^{1+\nu} \right] + \Psi_0(\mu_z, 0)$$

Consider now an economy with $\pi = \pi^\star$ and another one with $\pi = \tilde{\pi} \neq \pi^\star$. Imagine that in the latter, consumer are compensated in consumption units in such a way that they are as well off with $\tilde{\pi}$ as with $\pi^\star$. Let $1 + \varphi(\pi)$ denote this percentage increase in consumption. Thus $\varphi(\pi)$ is such that

$$\begin{aligned} \mathscr{W}(\pi^\star) &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \log\left((1+\varphi)(1-\eta)\hat{C}(\pi)\right) - \frac{\chi}{1+\nu} N_n^{1+\nu} \right] + \Psi_0(\mu_z, 0) \\ &= \frac{\log(1+\varphi(\pi))}{1-\beta} + \mathscr{W}(\pi) \end{aligned}$$

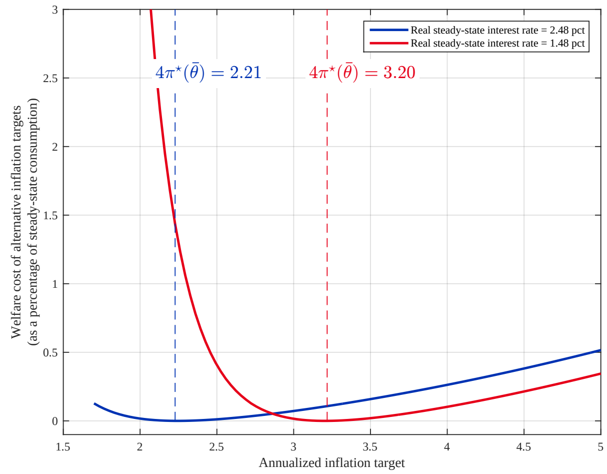
It then follows that

$$\varphi(\pi) = \exp\{(1-\beta)[\mathscr{W}(\pi^\star) - \mathscr{W}(\pi)]\} - 1.$$

In practice, welfare is approximated to second order.

We compute $\varphi(\pi)$ under two alternative steady-state interest rate scenarios. In the first scenario, we set $r^\star$ to the baseline estimated value, corresponding to the posterior mean of $\mu_z + \rho$. In the second scenario, we consider a downward shift in $\mu_z$ by one percentage point (in annual terms), resulting in a lower steady-state real rate. The results are reported in Figure F.1. The blue lines show $\varphi(\pi)$ in the first scenario and the red lines show $\varphi(\pi)$ under a lower real interest rate. For ease of interpretation, the dashed, vertical lines indicate the optimal values of inflation under the two alternative interest rate scenarios.

Figure F.1: Welfare cost of inflation at the posterior mean



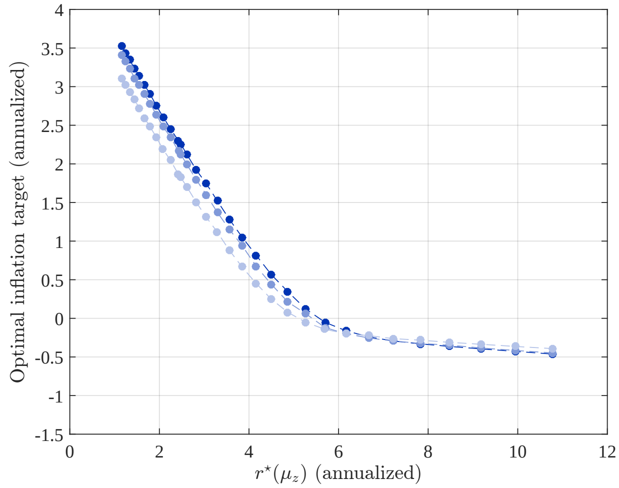

**Note:** The figure reports the welfare cost of inflation stated as a percentage of steady-state consumption in the optimal setting.

Figures F.1 suggests that in the baseline scenario, the welfare cost of raising or lowering the inflation target by one percentage point is relatively mild. However, this conclusion is not robust to a lower real interest rate. As the red line shows, with a one percentage rate lower $r^{\star}$, the welfare cost of inflation is asymmetric. It would be much costlier to lower the inflation target than to raise it in the neighborhood of the optimal target. In particular, keeping the inflation target unchanged when faced with a one-percentage point decline in $r^{\star}$ give rise to a 1.5% consumption loss.

# G Further illustrations of the $(r^\star, \pi^\star)$ relation

## G.1 When $\mu_z$ varies

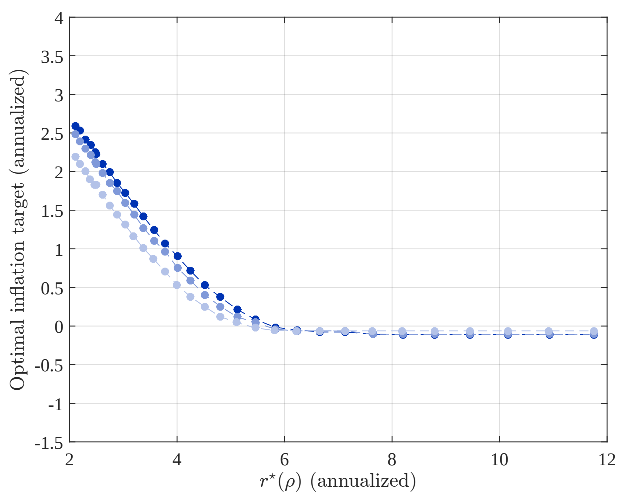
Figure G.1: $(r^\star, \pi^\star)$ locus when $\mu_z$ varies




**Note:** Blue: parameters set at the posterior mean; light blue: parameters set at the posterior median; Lighter blue: parameters set at the posterior mode. Memo: $r^\star = \rho + \mu_z$. Range for $\mu_z$: 0.4% to 10% (annualized) .

## G.2 When $\rho$ varies

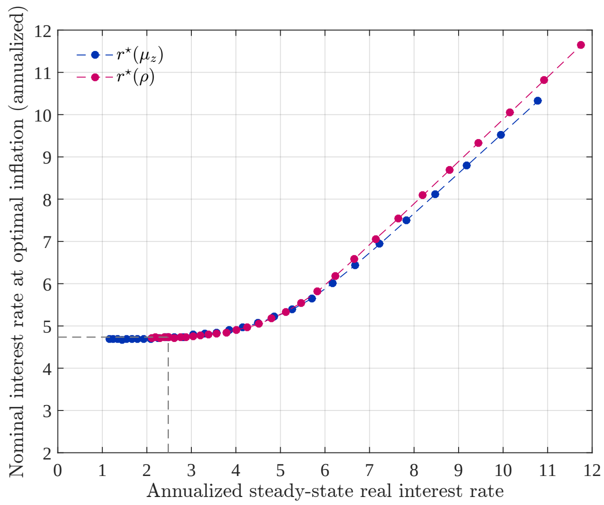
Figure G.2: $(r^\star, \pi^\star)$ locus when $\rho$ varies




**Note:** Blue: parameters set at the posterior mean; light blue: parameters set at the posterior median; Lighter blue: parameters set at the posterior mode. Memo: $r^\star = \rho + \mu_z$. Range for $\mu_z$: 0.4% to 10% (annualized) .

# H Nominal and Real Interest Rates

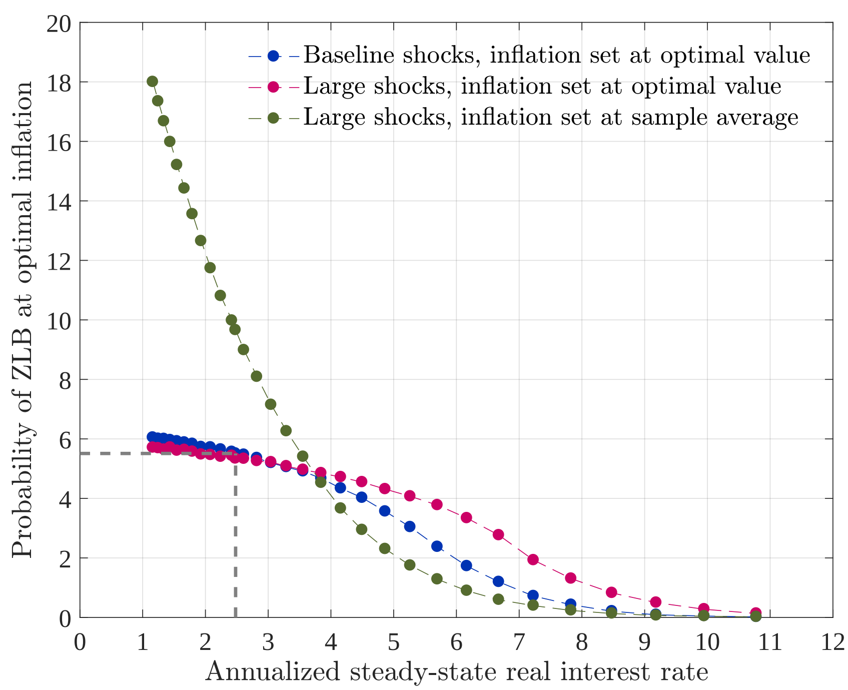
Figure H.1: $(r^\star, i^\star)$ locus (at the posterior mean)




**Note:** the blue dots correspond to the $(r^\star, i^\star)$ locus when $r^\star$ varies with $\mu_z$; the red dots correspond to the $(r^\star, i^\star)$ locus when $r^\star$ varies with $\rho$

# I The probability of ZLB under large shocks

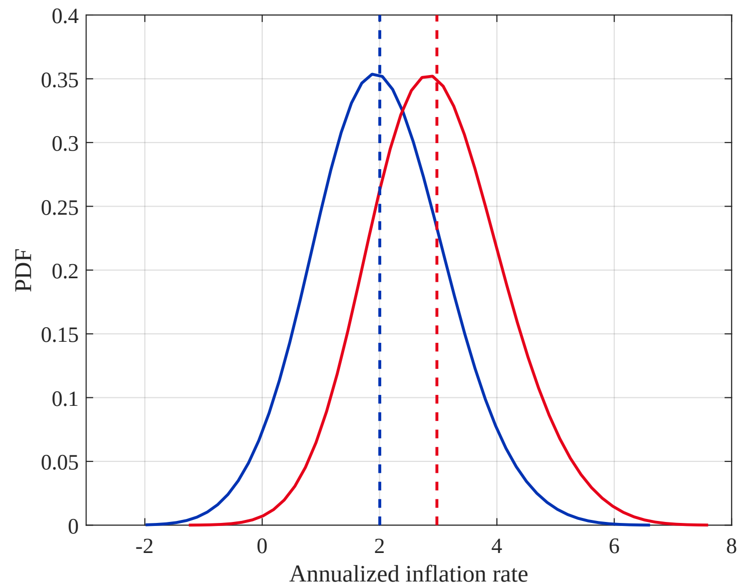
Figure I.1: Relation between probability of ZLB at optimal inflation and $r^\star$ (at the posterior mean)




**Note:** the blue dots correspond to the $(r^\star, \pi^\star)$ locus when $r^\star$ varies with $\mu_z$ in the baseline; the red dots correspond to the locus in the "large shocks" case; the green dots correspond to the $(r^\star, \pi^\star)$ locus in the "large shocks" case when $\pi^\star$ is left at its sample inflation value.

# J Distribution of $\pi^\star$ following a downward shift of the distribution of $r^\star$

Figure J.1: Counterfactual - US




**Note:** The dashed vertical line indicates the mean value, i.e. $\mathbb{E}_\theta(\pi^\star(\theta))$.

# K Model Solution

## K.1 Households

### K.1.1 First Order Conditions

The Lagrangian associated with the program (1) under constraint (2) is

$$\mathscr{L}_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Bigg\{ \mathrm{e}^{\zeta_{c,t+s}} \log(C_{t+s} - \hat{\eta} C_{t+s-1}) - \frac{\chi}{1+\nu} \int_0^1 \mathrm{e}^{\zeta_{h,t+s}} (N_{t+s}(h))^{1+\nu} \mathrm{d}h$$
$$- \frac{\Lambda_{t+s}}{P_{t+s}} \left[ P_{t+s} C_{t+s} + Q_{t+s} B_{t+s} \mathrm{e}^{-\zeta_{q,t+s}} + P_{t+s} \mathrm{tax}_{t+s} - \int_0^1 W_{t+s}(h) N_{t+s}(h) \mathrm{d}h - B_{t+s-1} - P_{t+s} \mathrm{div}_{t+s} \right] \Bigg\},$$

The associated first-order condition with respect to bonds is

$$\frac{\partial \mathscr{L}_t}{\partial B_t} = 0 \Leftrightarrow \Lambda_t Q_t \mathrm{e}^{-\zeta_{q,t}} = \beta \mathbb{E}_t \left\{ \frac{\Lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{K.1}$$

and the first-order condition with respect to consumption is

$$\frac{\partial \mathscr{L}_t}{\partial C_t} = 0 \Leftrightarrow \frac{\mathrm{e}^{\zeta_{c,t}}}{C_t - \hat{\eta} C_{t-1}} - \beta \hat{\eta} \mathbb{E}_t \left\{ \frac{\mathrm{e}^{\zeta_{c,t+1}}}{C_{t+1} - \eta C_t} \right\} = \Lambda_t. \tag{K.2}$$

where $\Pi_t \equiv P_t / P_{t-1}$ represents the (gross) inflation rate, and

We induce stationarity by normalizing trending variables by the level of technical progress. To this end, we use the subscript $z$ to refer to a normalized variable. For example, we define

$$C_{z,t} \equiv \frac{C_t}{Z_t}, \qquad \Lambda_{z,t} \equiv \Lambda_t Z_t,$$

where it is recalled that

$$Z_t = \mathrm{e}^{z_t}$$

with

$$z_t = \mu_z + z_{t-1} + \zeta_{z,t}.$$

We then rewrite the first order condition in terms of the normalized variables. Equation (K.2) thus rewrites

$$\frac{\mathrm{e}^{\zeta_{c,t}}}{C_{z,t} - \eta C_{z,t-1}\mathrm{e}^{-\zeta_{z,t}}} - \beta\eta\mathbb{E}_t\left\{\mathrm{e}^{-\zeta_{z,t+1}}\frac{\mathrm{e}^{\zeta_{c,t+1}}}{C_{z,t+1} - \eta C_{z,t}\mathrm{e}^{-\zeta_{z,t+1}}}\right\} = \Lambda_{z,t}, \tag{K.3}$$

Similarly, equation (K.1) rewrites

$$\Lambda_{z,t} Q_t \mathrm{e}^{-\zeta_{q,t}} = \beta\mathrm{e}^{-\mu_z}\mathbb{E}_t\left\{\mathrm{e}^{-\zeta_{z,t+1}}\frac{\Lambda_{z,t+1}}{\Pi_{t+1}}\right\}, \tag{K.4}$$

where we defined

$$\eta \equiv \hat{\eta}\mathrm{e}^{-\mu_z}.$$

Let us define $i_t \equiv -\log(Q_t)$ and for any generic variable $X_t$

$$x_t \equiv \log(X_t), \quad \hat{x}_t \equiv x_t - x$$

where $x$ is the steady-state value of $x$. Using these definitions, log-linearizing equation (K.3) yields

$$\hat{g}_t + \beta\eta\mathbb{E}_t\{\hat{c}_{t+1}\} - (1+\beta\eta^2)\hat{c}_t + \eta\hat{c}_{t-1} - \eta(\zeta_{z,t} - \beta\mathbb{E}_t\{\zeta_{z,t+1}\}) = \varphi^{-1}\hat{\lambda}_t \tag{K.5}$$

where we defined

$$\varphi^{-1} \equiv (1-\beta\eta)(1-\eta),$$

$$\hat{g}_t = (1-\eta)(\zeta_{c,t} - \beta\eta\mathbb{E}_t\{\zeta_{c,t+1}\}).$$

Similarly, log-linearizing equation (K.4) yields

$$\hat{\lambda}_t = \hat{\imath}_t + \mathbb{E}_t\{\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} - \zeta_{z,t+1}\} + \zeta_{q,t}. \tag{K.6}$$

## K.2 Firms

Expressing the demand function in normalized terms yields

$$Y_{z,t}(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\theta_p} Y_{z,t},$$

In the case of a firm not drawn to re-optimize, this equation specializes to (in log-linear terms)

$$\hat{y}_{t,t+s}(f) - \hat{y}_{t+s} = \theta_p(\hat{\pi}_{t,t+s} - \hat{\delta}^p_{t,t+s} - \hat{p}^\star_t(f)). \tag{K.7}$$

### K.2.1 Cost Minimization

The real cost of producing $Y_t(f)$ units of good of $f$ is

$$\frac{W_t}{P_t} L_t(f) = \frac{W_t}{P_t} \left( \frac{Y_t(f)}{Z_t} \right)^{\phi} \tag{K.8}$$

The associated real marginal cost is thus

$$S_t(f) = \phi \frac{W_t}{P_t Z_t} \left( \frac{Y_t(f)}{Z_t} \right)^{\phi - 1} \tag{K.9}$$

It is useful at this stage to restate the production function in log-linearized terms:

$$\hat{y}_{z,t}(f) = \frac{1}{\phi} \hat{n}_t(f) \tag{K.10}$$

### K.2.2 Price Setting of Intermediate Goods: Optimization

Firm $f$ chooses $P_t^\star(f)$ in order to maximize

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta \alpha_p)^s \Lambda_{t+s} \left\{ (1 + \tau_{p,t+s}) \frac{V_{t,t+s}^p P_t^\star(f)}{P_{t+s}} Y_{t,t+s}^\star(f) - S\left(Y_{t,t+s}(f)\right) \right\}, \tag{K.11}$$

subject to the demand function

$$Y_{t,t+s}^\star(f) = \left( \frac{V_{t,t+s}^p P_t^\star(f)}{P_{t+s}} \right)^{-\theta_p} Y_{t+s}.$$

and the cost schedule (K.8), where $\Lambda_t$ is the representative household's marginal utility of wealth, and $\mathbb{E}_t\{\cdot\}$ is the expectation operator conditional on information available as of time $t$. That $\Lambda_t$ appears in the above maximization program reflects the fact that the representative household is the ultimate owner of firm $f$.

The associated first-order condition is

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta \alpha_p)^s \Lambda_{t+s} \left\{ \left( \frac{V_{t,T}^p P_t^\star(f)}{P_{t+s}} \right)^{1-\theta_p} Y_{t+s} - \frac{\mu_p}{1+\tau_p} \mathrm{e}^{\zeta_{u,t+s}} \frac{W_{t+s}}{P_{t+s}} \phi \left( \left( \frac{V_{t,t+s}^p P_t^\star(f)}{P_{t+s}} \right)^{-\theta_p} \frac{Y_{t+s}}{Z_{t+s}} \right)^{\phi} \right\} = 0,$$

where

$$\mu_p \equiv \frac{\theta_p}{\theta_p - 1}.$$

This rewrites

$$\left( \frac{P_t^\star(f)}{P_t} \right)^{1+\theta_p(\phi-1)} = \frac{\mu_p}{1+\tau_p} \frac{K_{p,t}}{F_{p,t}}$$

where

$$K_{p,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \alpha_p)^s \Lambda_{z,t+s} \mathrm{e}^{\zeta_{u,t+s}} \frac{W_{z,t+s}}{P_{t+s}} \phi \left( \left( \frac{V_{t,t+s}^p}{\Pi_{t,t+s}} \right)^{-\theta_p} Y_{z,t+s} \right)^{\phi}$$

and

$$F_{p,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\alpha_p)^s \Lambda_{z,T} \left( \frac{V^p_{t,t+s}}{\Pi_{t,t+s}} \right)^{1-\theta_p} Y_{z,t+s},$$

where $\Pi_{t,t+s} \equiv P_{t+s}/P_t$.

Notice that

$$K_{p,t} = \phi \Lambda_{z,t} e^{\zeta_{u,t}} \frac{W_{z,t}}{P_t} (Y_{z,t})^{\phi} + \beta\alpha_p \mathbb{E}_t \left( \frac{(\Pi_t))^{\gamma_p}}{\Pi_{t+1}} \right)^{-\phi\theta_p} K_{p,t+1},$$

and

$$F_{p,t} = \Lambda_{z,t} Y_{z,t} + \beta\alpha_p \mathbb{E}_t \left( \frac{(\Pi_t)^{\gamma_p}}{\Pi_{t+1}} \right)^{1-\theta_p} F_{p,t+1}.$$

With a slight abuse of notation, we obtain the steady-state relation

$$\left( \frac{P^\star}{P} \right)^{1+\theta_p(\phi-1)} = \frac{\mu_p}{1+\tau_p} \phi \frac{W_z}{P} Y_z^{\phi-1} \frac{1-\beta\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}}{1-\beta\alpha_p(\Pi)^{\phi\theta_p(1-\gamma_p)}}.$$

Log-linearizing yields

$$[1+\theta_p(\phi-1)](p_t^\star - p_t) = \hat{k}_{p,t} - \hat{f}_{p,t}$$

$$\hat{k}_{p,t} = (1-\omega_{K,p})[\hat{\lambda}_{z,t} + \hat{\omega}_t + \phi\hat{y}_{z,t} + \zeta_{u,t}] + \omega_{K,p}\mathbb{E}_t\{\hat{k}_{p,t+1} + \phi\theta_p(\hat{\pi}_{t+1} - \gamma_p\hat{\pi}_t)\},$$

and

$$\hat{f}_{p,t} = (1-\omega_{F,p})(\hat{\lambda}_{z,t} + \hat{y}_{z,t}) + \omega_{F,p}\mathbb{E}_t\{\hat{f}_{p,t+1} + (\theta_p - 1)(\hat{\pi}_{t+1} - \gamma_p\hat{\pi}_t)\}.$$

where we defined the de-trended real wage

$$\omega_t \equiv w_{z,t} - p_t$$

and the auxiliary parameters

$$\omega_{K,p} \equiv \beta\alpha_p(\Pi)^{(1-\gamma_p)\phi\theta_p}$$

and

$$\omega_{F,p} \equiv \beta\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}.$$

Finally, notice that

$$\begin{aligned} P_t^{1-\theta_p} &= \int_0^1 P_t(f)^{1-\theta_p} \mathrm{d}f \\ &= (1-\alpha_p)(P_t^\star)^{1-\theta_p} + \alpha_p \int_0^1 \left[ (\Pi_{t-1}))^{\gamma_p} P_{t-1}(f) \right]^{1-\theta_p} \mathrm{d}f. \end{aligned}$$

Thus

$$1 = (1-\alpha_p) \left( \frac{P_t^\star}{P_t} \right)^{1-\theta_p} + \alpha_p \left[ \frac{(\Pi_{t-1})^{\gamma_p}}{\Pi_t} \right]^{1-\theta_p}.$$

The steady-state relation is

$$\left( \frac{P^\star}{P} \right)^{1-\theta_p} = \frac{1-\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}}{1-\alpha_p}.$$

Log-linearizing this yields

$$\hat{p}_t^\star = \frac{\omega_{F,p}}{\beta - \omega_{F,p}} (\hat{\pi}_t - \gamma_p \hat{\pi}_{t-1}).$$

### K.3 Unions

#### K.3.1 Wage Setting

Union $h$ sets $W_t^\star(h)$ so as to maximize

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\alpha_w)^s \left\{ (1+\tau_w) \frac{\Lambda_{t+s}}{P_{t+s}} \mathrm{e}^{\gamma_z \mu_z s} V_{t,t+s}^w W_t^\star(h) N_{t,t+s}(h) - \frac{\chi}{1+\nu} \mathrm{e}^{\zeta_{h,t+s}} (N_{t,t+s}(h))^{1+\nu} \right\},$$

where

$$N_{t,t+s}(h) = \left( \frac{\mathrm{e}^{\gamma_z \mu_z s} V_{t,t+s}^w W_t^\star(h)}{W_{t+s}} \right)^{-\theta_w} N_{t+s}.$$

The associated first-order condition is

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\alpha_w)^s \Bigg\{ \Lambda_T \frac{W_{t+s}}{P_{t+s}} h_{t+s} \left( \frac{\mathrm{e}^{\gamma_z \mu_z s} V_{t,T}^w}{\Pi_{t,t+s}^w} \frac{W_t^\star(h)}{W_{t+s}} \right)^{1-\theta_w} - \frac{\mu_w}{1+\tau_w} \chi \mathrm{e}^{\zeta_{h,t+s}} \left( \frac{\mathrm{e}^{\gamma_z \mu_z s} V_{t,t+s}^w}{\Pi_{t,t+s}^w} \frac{W_t^\star(h)}{W_{t+s}} \right)^{-(1+\nu)\theta_w} N_{t+s}^{1+\nu} \Bigg\} = 0,$$

where $\Pi_{t,t+s}^w = W_{t+s}/W_t$.

Rearranging yields

$$\left( \frac{W_t^\star(h)}{W_t} \right)^{1+\theta_w \nu} = \frac{\mu_w}{1+\tau_w} \frac{K_{w,t}}{F_{w,t}},$$

where

$$K_{w,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\alpha_w)^s \left\{ \chi \mathrm{e}^{\zeta_{h,t+s}} \left( \frac{\mathrm{e}^{\gamma_z \mu_z s} V_{t,t+s}^w}{\Pi_{t,t+s}^w} \right)^{-(1+\nu)\theta_w} N_{t+s}^{1+\nu} \right\},$$

$$F_{w,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\alpha_w)^s \left\{ \Lambda_{t+s} \frac{W_{t+s}}{P_{t+s}} N_{t+s} \left( \frac{\mathrm{e}^{\gamma_z \mu_z s} V_{t,t+s}^w}{\Pi_{t,t+s}^w} \right)^{1-\theta_w} \right\},$$

and where $\Pi_{t,t+s}^w \equiv W_{t+s}/W_t$.

Notice that

$$K_{w,t} = \chi \mathrm{e}^{\zeta_{h,t}} N_t^{1+\nu} + \beta\alpha_w \mathbb{E}_t \left\{ \left( \mathrm{e}^{\gamma_z \mu_z} \frac{(\Pi_t)^{\gamma_w}}{\Pi_{w,t+1}} \right)^{-(1+\nu)\theta_w} K_{w,t+1} \right\},$$

and

$$F_{w,t} = \Lambda_{z,t} \frac{W_{z,t}}{P_t} N_t + \beta\alpha_w \mathbb{E}_t \left\{ \left( \mathrm{e}^{\gamma_z \mu_z} \frac{(\Pi_t)^{\gamma_w}}{\Pi_{w,t+1}} \right)^{1-\theta_w} F_{w,t+1} \right\}.$$

The associated steady-state relations are

$$\left( \frac{W^\star}{W} \right)^{1+\theta_w \nu} = \frac{\mu_w}{1+\tau_w} \frac{K_w}{F_w},$$

$$K_w = \frac{\chi N^{1+\nu}}{1 - \beta\alpha_w [\mathrm{e}^{(1-\gamma_z)\mu_z} (\Pi)^{1-\gamma_w}]^{(1+\nu)\theta_w}},$$

$$F_w = \frac{\Lambda \frac{W_z}{P} H}{1 - \beta \alpha_w [\mathrm{e}^{(1-\gamma_z)\mu_z} (\Pi)^{1-\gamma_w}]^{\theta_w - 1}}.$$

Log-linearizing the above equations finally yields

$$(1 + \theta_w \nu)(w_t^\star - w_t) = \hat{k}_{w,t} - \hat{f}_{w,t},$$

$$\hat{k}_{w,t} = (1 - \omega_{K,w})[(1+\nu)\hat{n}_t + \zeta_{h,t}] + \omega_{K,w} \mathbb{E}_t \{\hat{k}_{w,t+1} + (1+\nu)\theta_w (\hat{\pi}_{w,t+1} - \gamma_w \hat{\pi}_t)\},$$

$$\hat{f}_{w,t} = (1 - \omega_{F,w})(\hat{\lambda}_{z,t} + \hat{\omega}_t + \hat{n}_t) + \omega_{F,w} \mathbb{E}_t \{\hat{f}_{w,t+1} + (\theta_w - 1)(\hat{\pi}_{w,t+1} - \gamma_w \hat{\pi}_t)\},$$

where we defined

$$\omega_{K,w} = \beta \alpha_w [\mathrm{e}^{(1-\gamma_z)\mu_z} (\Pi)^{(1-\gamma_w)}]^{(1+\nu)\theta_w},$$

$$\omega_{F,w} = \beta \alpha_w [\mathrm{e}^{(1-\gamma_z)\mu_z} (\Pi)^{(1-\gamma_w)}]^{\theta_w - 1}.$$

To complete this section, notice that

$$1 = (1 - \alpha_w) \left( \frac{W_t^\star}{W_t} \right)^{1-\theta_w} + \alpha_w \left( \mathrm{e}^{\gamma_z \mu_z} \frac{[\Pi_{t-1}]^{\gamma_w}}{\Pi_{w,t}} \right)^{1-\theta_w}$$

and

$$w_t^\star - w_t = \frac{\omega_{F,w}}{\beta - \omega_{F,w}} (\hat{\pi}_{w,t} - \gamma_w \hat{\pi}_{t-1}).$$

### K.4 Market Clearing

The clearing on the labor market implies

$$N_t = \left( \frac{Y_t}{Z_t} \right)^{\phi} \int_0^1 \left( \frac{P_t(f)}{P_t} \right)^{-\phi\theta_p} \mathrm{d}f.$$

Let us define

$$\Xi_{p,t} = \left( \int_0^1 \left( \frac{P_t(f)}{P_t} \right)^{-\phi\theta_p} \mathrm{d}f \right)^{-1/(\phi\theta_p)},$$

so that

$$N_t = (Y_{z,t} \Xi_{p,t}^{-\theta_p})^{\phi}.$$

Hence, expressed in log-linear terms, this equation reads

$$\hat{n}_t = \phi(\hat{y}_{z,t} - \theta_p \hat{\xi}_{p,t}).$$

Notice that

$$\Xi_{p,t}^{-\phi\theta_p} = (1-\alpha_p)\left(\frac{P_t^\star}{P_t}\right)^{-\phi\theta_p} + \alpha_p\left(\frac{[\Pi_{t-1}]^{\gamma_p}}{\Pi_t}\right)^{-\phi\theta_p}\Xi_{p,t-1}^{-\phi\theta_p}.$$

The associated steady-state relation is

$$\Xi_p^{-\phi\theta_p} = \frac{(1-\alpha_p)}{1-\alpha_p(\Pi)^{(1-\gamma_p)\phi\theta_p}}\left(\frac{P^\star}{P}\right)^{-\phi\theta_p}.$$

Log-linearizing the price dispersion yields

$$\hat{\xi}_{p,t} = (1-\omega_\Xi)(p_t^\star - p_t) + \omega_\Xi[\hat{\xi}_{p,t-1} - (\hat{\pi}_t - \gamma_p\hat{\pi}_{t-1})]$$

where we defined

$$\omega_\Xi = \alpha_p(\Pi)^{(1-\gamma_p)\phi\theta_p}.$$

## K.5 Natural Rate of Output

The natural rate of output is the level of production that would prevail in an economy without nominal rigidities, i.e. $\alpha_p = \alpha_w = 0$ and without cost-push shocks (i.e., $\zeta_{u,t} = 0$). Under such circumstances, the dynamic system simplifies to

$$\hat{w}_{z,t}^n + (\phi-1)\hat{y}_{z,t}^n = 0,$$

$$\nu\hat{n}_t^n + \zeta_{h,t} = \hat{\lambda}_{z,t}^n + \hat{w}_{z,t}^n,$$

$$\hat{n}_t^n = \phi\hat{y}_{z,t}^n,$$

$$\hat{g}_t + \beta\eta\mathbb{E}_t\{\hat{y}_{z,t+1}^n\} - (1+\beta\eta^2)\hat{y}_{z,t}^n + \eta\hat{y}_{z,t-1}^n - \eta(\zeta_{z,t} - \beta\mathbb{E}_t\{\zeta_{z,t+1}\}) = \varphi^{-1}\hat{\lambda}_{z,t}^n,$$

where the superscript $n$ stands for *natural*.

Combining these equations yields

$$[\varphi(1+\beta\eta^2)+\omega]\hat{y}_{z,t}^n - \varphi\beta\eta\mathbb{E}_t\{\hat{y}_{z,t+1}^n\} - \varphi\eta\hat{y}_{z,t-1}^n = \varphi\hat{g}_t - \zeta_{h,t} - \varphi\eta\zeta_{z,t}^*$$

where we defined

$$\omega \equiv \nu\phi + \phi - 1,$$

and

$$\zeta_{z,t}^* = \zeta_{z,t} - \beta\mathbb{E}_t\{\zeta_{z,t+1}\}$$

## K.6 Working Out the Steady State

The steady state is defined by the following set of equations

$$\frac{1-\beta\eta}{(1-\eta)C} = \Lambda_z,$$

$$e^{-i} = \beta e^{-\mu_z}\Pi^{-1},$$

$$\left(\frac{P^\star}{P}\right)^{1+\theta_p(\phi-1)} = \frac{\mu_p}{1+\tau_p}\frac{K_p}{F_p},$$

$$K_p = \frac{\phi\Lambda_z \frac{W_z}{P} Y_z^{\phi}}{1-\beta\alpha_p(\Pi)^{\phi\theta_p(1-\gamma_p)}},$$

$$F_p = \frac{\Lambda_z Y_z}{1-\beta\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}},$$

$$\left(\frac{P^\star}{P}\right)^{1-\theta_p} = \frac{1-\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}}{1-\alpha_p},$$

$$\left(\frac{W^\star}{W}\right)^{1+\theta_w\nu} = \frac{\mu_w}{1+\tau_w}\frac{K_w}{F_w},$$

$$K_w = \frac{\chi N^{1+\nu}}{1-\beta\alpha_w[e^{(1-\gamma_z)\mu_z}(\Pi)^{1-\gamma_w}]^{(1+\nu)\theta_w}},$$

$$F_w = \frac{\Lambda_z \frac{W_z}{P} H}{1-\beta\alpha_w[e^{(1-\gamma_z)\mu_z}(\Pi)^{1-\gamma_w}]^{\theta_w-1}},$$

$$\left(\frac{W^\star}{W}\right)^{1-\theta_w} = \frac{1-\alpha_w[e^{(1-\gamma_z)\mu_z}(\Pi)^{(1-\gamma_w)}]^{\theta_w-1}}{1-\alpha_w},$$

$$\Pi_w = \Pi e^{\mu_z}$$

We can solve for $i$ and $\Pi_w$ using

$$\Pi_w = \Pi e^{\mu_z}$$

$$1 = \beta e^{-\mu_z} e^{i}\Pi^{-1},$$

Standard manipulations yield

$$\frac{1-\omega_{K,p}}{1-\omega_{F,p}}\left(\frac{\beta(1-\alpha_p)}{\beta-\omega_{F,p}}\right)^{\frac{1+\theta_p(\phi-1)}{\theta_p-1}} = \frac{\mu_p}{1+\tau_p}\phi\frac{W_z}{P}Y_z^{\phi-1},$$

where we used

$$\omega_{K,p} = \beta\alpha_p(\Pi)^{(1-\gamma_p)\phi\theta_p}$$

$$\omega_{F,p} = \beta\alpha_p(\Pi)^{(1-\gamma_p)(\theta_p-1)}$$

Similar manipulations yield

$$\frac{1-\omega_{K,w}}{1-\omega_{F,w}}\left(\frac{\beta(1-\alpha_w)}{\beta-\omega_{F,w}}\right)^{\frac{1+\theta_w\nu}{\theta_w-1}} = \frac{\mu_w}{1+\tau_w}\frac{\chi N^\nu}{\Lambda_z\frac{W_z}{P}},$$

where we used

$$\omega_{K,w} = \beta\alpha_w[\mathrm{e}^{(1-\gamma_z)\mu_z}(\Pi)^{(1-\gamma_w)}]^{(1+\nu)\theta_w}$$

$$\omega_{F,w} = \beta\alpha_w[\mathrm{e}^{(1-\gamma_z)\mu_z}(\Pi)^{(1-\gamma_w)}]^{\theta_w-1}$$

Combining these conditions yields

$$\frac{1-\omega_{K,w}}{1-\omega_{F,w}}\left(\frac{\beta(1-\alpha_w)}{\beta-\omega_{F,w}}\right)^{\frac{1+\theta_w\nu}{\theta_w-1}}\frac{1-\omega_{K,p}}{1-\omega_{F,p}}\left(\frac{\beta(1-\alpha_p)}{\beta-\omega_{F,p}}\right)^{\frac{1+\theta_p(\phi-1)}{\theta_p-1}} = \frac{\mu_w}{1+\tau_w}\frac{\mu_p}{1+\tau_p}\frac{1-\eta}{1-\beta\eta}\phi\chi N^\nu Y_z^\phi$$

Now, recall that

$$(Y_z\Xi_p^{-\theta_p})^\phi = N$$

Then, using

$$\Xi_p^{-\phi\theta_p} = \frac{1-\alpha_p}{1-\omega_\Xi}\left(\frac{P^\star}{P}\right)^{-\phi\theta_p},$$

and

$$\left(\frac{P^\star}{P}\right)^{-\phi\theta_p} = \left(\frac{\beta(1-\alpha_p)}{\beta-\omega_{F,p}}\right)^{-\phi\frac{\theta_p}{\theta_p-1}}$$

we end up with

$$N^\nu Y_z^\phi = \left(\frac{1-\alpha_p}{1-\omega_\Xi}\left(\frac{\beta(1-\alpha_p)}{\beta-\omega_{F,p}}\right)^{-\phi\frac{\theta_p}{\theta_p-1}}\right)^\nu Y_z^{(1+\nu)\phi},$$

so that

$$\Omega = \frac{\mu_w}{1+\tau_w}\frac{\mu_p}{1+\tau_p}\frac{1-\eta}{1-\beta\eta}\phi\chi Y_z^{(1+\nu)\phi},$$

where

$$\Omega = \frac{1-\omega_{K,w}}{1-\omega_{F,w}}\left(\frac{\beta(1-\alpha_w)}{\beta-\omega_{F,w}}\right)^{\frac{1+\theta_w\nu}{\theta_w-1}}\frac{1-\omega_{K,p}}{1-\omega_{F,p}}\left(\frac{\beta(1-\alpha_p)}{\beta-\omega_{F,p}}\right)^{\frac{1+\theta_p[(1+\nu)\phi-1]}{\theta_p-1}}\left(\frac{1-\omega_\Xi}{1-\alpha_p}\right)^\nu$$

Recall that we defined the natural rate of output as the level of production that would prevail in an economy without nominal rigidities, i.e. $\alpha_p = \alpha_w = 0$, and no cost-push shock. Under such circumstances, the steady-state value of the (normalized) natural rate of output $Y_z^n$ obeys

$$1 = \frac{\mu_w}{1+\tau_w}\frac{\mu_p}{1+\tau_p}\frac{1-\eta}{1-\beta\eta}\phi\chi(Y_z^n)^{\phi(1+\nu)}.$$

It follows that the steady-state distortion due to sticky prices and wages (and less than perfect indexation) is

$$\left(\frac{Y_z}{Y_z^n}\right)^{\phi(1+\nu)} = \Omega.$$

# L Welfare

Let us define for any generic variable $X_t$

$$\frac{X_t - X}{X} = \hat{x}_t + \frac{1}{2}\hat{x}_t^2 + \mathcal{O}(||\zeta||^3)$$

$$\frac{X_t - X^n}{X^n} = \tilde{x}_t + \frac{1}{2}\tilde{x}_t^2 + \mathcal{O}(||\zeta||^3)$$

Below, we repeatedly use the following two lemmas:

**Lemma 1.** *Let $g(\cdot)$ be a twice differentiable function and let $X$ be a stationary random variable. Then*

$$\mathbb{E}\{g(X)\} = g(\mathbb{E}\{X\}) + \frac{1}{2}g''(\mathbb{E}\{X\})\mathbb{V}\{X\} + \mathcal{O}(||X||^3).$$

**Lemma 2.** *Let $g(\cdot)$ be a twice differentiable function and let $x$ be a stationary random variable. Then*

$$\mathbb{V}\{g(X)\} = [g'(\mathbb{E}\{X\})]^2\mathbb{V}\{X\} + \mathcal{O}(||X||^3).$$

In the rest of this section, we take a second-order approximation of welfare, where we consider the inflation rate as an expansion parameter. It follows that we consider the welfare effects of non-zero trend inflation only up to second order.

## L.1 Second-Order Approximation of Utility

Consider first the utility derived from consumption. For the sake of notational simplicity, define

$$U(C_{z,t} - \eta C_{z,t-1}\mathrm{e}^{-\zeta_{z,t}}) = \log(C_{z,t} - \eta C_{z,t-1}\mathrm{e}^{-\zeta_{z,t}})$$

We thus obtain

$$\begin{aligned}
\mathrm{e}^{\zeta_{c,t}}U(C_{z,t} - \eta C_{z,t-1}\mathrm{e}^{-\zeta_{z,t}}) = \frac{1}{1-\eta}\Bigg[ & \left(\frac{C_{z,t} - C_z^n}{C_z^n}\right) - \eta\left(\frac{C_{z,t-1} - C_z^n}{C_z^n}\right) \\
& - \frac{1}{2}\frac{1}{(1-\eta)}\left(\frac{C_{z,t} - C_z^n}{C_z^n}\right)^2 + \frac{\eta}{(1-\eta)}\left(\frac{C_{z,t} - C_z^n}{C_z^n}\right)\left(\frac{C_{z,t-1} - C_z^n}{C_z^n}\right) - \frac{1}{2}\frac{\eta^2}{(1-\eta)}\left(\frac{C_{z,t-1} - C_z^n}{C_z^n}\right)^2 \\
& + \zeta_{c,t}\left(\frac{C_{z,t} - C_z^n}{C_z^n}\right) - \eta\zeta_{c,t}\left(\frac{C_{z,t-1} - C_z^n}{C_z^n}\right) - \frac{\eta}{(1-\eta)}\zeta_{z,t}\left(\frac{C_{z,t} - C_z^n}{C_z^n}\right) + \frac{\eta}{(1-\eta)}\zeta_{z,t}\left(\frac{C_{z,t-1} - C_z^n}{C_z^n}\right)\Bigg] \\
& + \text{t.i.p} + \mathcal{O}(||\zeta||^3),
\end{aligned}$$

where t.i.p stands for terms independent of policy.

Then, using

$$\frac{C_{z,t} - C_z^n}{C_z^n} = \tilde{c}_{z,t} + \frac{1}{2}\tilde{c}_{z,t}^2 + \mathcal{O}(||\zeta||^3)$$

we obtain

$$\begin{aligned} e^{\zeta_{c,t}} U(C_{z,t} - \eta C_{z,t-1} e^{-\zeta_{z,t}}) = \frac{1}{1-\eta} \Bigg[ & \tilde{c}_{z,t} - \eta \tilde{c}_{z,t-1} + \frac{1}{2}(\tilde{c}_{z,t}^2 - \eta \tilde{c}_{z,t-1}^2) \\ & - \frac{1}{2}\frac{1}{1-\eta}\tilde{c}_{z,t}^2 + \frac{\eta}{1-\eta}\tilde{c}_{z,t}\tilde{c}_{z,t-1} - \frac{1}{2}\eta^2 \frac{1}{1-\eta}\tilde{c}_{z,t-1}^2 \\ & + \zeta_{c,t}(\tilde{c}_{z,t} - \eta \tilde{c}_{z,t-1}) - \frac{\eta}{1-\eta}\zeta_{z,t}(\tilde{c}_{z,t} - \tilde{c}_{z,t-1}) \Bigg] + \text{t.i.p} + \mathcal{O}(||\zeta||^3), \end{aligned}$$

Using

$$\varphi^{-1} = (1 - \beta\eta)(1 - \eta)$$

we obtain

$$\begin{aligned} e^{\zeta_{c,t}} U(C_{z,t} - \eta C_{z,t-1} e^{-\zeta_{z,t}}) = \frac{1}{1-\eta} \Bigg[ & \tilde{y}_{z,t} - \eta \tilde{y}_{z,t-1} + \frac{1}{2}(\tilde{y}_{z,t}^2 - \eta \tilde{y}_{z,t-1}^2) \\ & - \frac{1}{2}(1-\beta\eta)\varphi \tilde{y}_{z,t}^2 + \eta(1-\beta\eta)\varphi \tilde{y}_{z,t}\tilde{y}_{z,t-1} - \frac{1}{2}\eta^2(1-\beta\eta)\varphi \tilde{y}_{z,t-1}^2 \\ & + \zeta_{c,t}(\tilde{y}_{z,t} - \eta \tilde{y}_{z,t-1}) - \eta(1-\beta\eta)\varphi \zeta_{z,t}(\tilde{y}_{z,t} - \tilde{y}_{z,t-1}) \Bigg] + \text{t.i.p} + \mathcal{O}(||\zeta||^3), \end{aligned}$$

where we imposed the equilibrium condition on the goods market.

Similarly, taking a second-order approximation of labor disutility in the neighborhood of the natural steady-state $N^n$ yields

$$\begin{aligned} \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} = \chi (N^n)^{1+\nu} \left( \frac{N_t(h) - N^n}{N^n} \right) + \frac{1}{2}\chi\nu (N^n)^{1+\nu} \left( \frac{N_t(h) - N^n}{N^n} \right)^2 \\ + \chi (N^n)^{1+\nu} \left( \frac{N_t(h) - N^n}{N^n} \right) \zeta_{h,t} + \text{t.i.p} + \mathcal{O}(||\zeta||^3). \end{aligned}$$

Now, using

$$\frac{N_t(h) - N^n}{N^n} = \tilde{n}_t(h) + \frac{1}{2}\tilde{n}_t(h)^2 + \mathcal{O}(||\zeta||^3)$$

we get

$$\frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} = \chi (N^n)^{1+\nu} \left[ \tilde{n}_t(h) + \frac{1}{2}(1+\nu)\tilde{n}_t(h)^2 + \tilde{n}_t(h)\zeta_{h,t} \right] + \text{t.i.p} + \mathcal{O}(||\zeta||^3).$$

Integrating over the set of labor types, one gets

$$\int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} \mathrm{d}h = \chi (N^n)^{1+\nu} \left[ \mathbb{E}_h\{\tilde{n}_t(h)\} + \frac{1}{2}(1+\nu)\mathbb{E}_h\{\tilde{n}_t(h)^2\} + \mathbb{E}_h\{\tilde{n}_t(h)\}\zeta_{h,t} \right] + \text{t.i.p} + \mathcal{O}(||\zeta||^3).$$

Now, since

$$\mathbb{V}_h\{\tilde{n}_t(h)\} = \mathbb{E}_h\{\tilde{n}_t(h)^2\} - \mathbb{E}_h\{\tilde{n}_t(h)\}^2$$

the above relation rewrites

$$\int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} dh = \chi (N^n)^{1+\nu} \Big[ \mathbb{E}_h\{\tilde{n}_t(h)\} + \frac{1}{2}(1+\nu)(\mathbb{V}_h\{\tilde{n}_t(h)\} + \mathbb{E}_h\{\tilde{n}_t(h)\}^2) \\ + \mathbb{E}_h\{\tilde{n}_t(h)\}\zeta_{h,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta||^3).$$

We need to express $\mathbb{E}_h\{\tilde{n}_t(h)\}$ and $\mathbb{V}_h\{\tilde{n}_t(h)\}$ in terms of the aggregate variables. To this end, we first establish a series of results, on which we draw later on.

## L.2 Aggregate Labor and Aggregate Output

Notice that

$$\frac{\theta_w - 1}{\theta_w} \tilde{n}_t = \log\left( \int_0^1 \left( \frac{N_t(h)}{N^n} \right)^{(\theta_w - 1)/\theta_w} dh \right).$$

Then, applying lemma 1, one obtains

$$\tilde{n}_t = \mathbb{E}_h\{\tilde{n}_t(h)\} + \frac{1}{2}\frac{\theta_w}{\theta_w - 1} \mathbb{E}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\}^{-2} \mathbb{V}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\} + \mathcal{O}(||\zeta||^3).$$

Then, notice that

$$\mathbb{V}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\} = \mathbb{V}_h\left\{ \exp\left[ (1 - \theta_w^{-1}) \log\left( \frac{N_t(h)}{N^n} \right) \right] \right\}$$

so that, by applying lemma 2, one obtains

$$\mathbb{V}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\} = (1 - \theta_w^{-1})^2 \exp\left( (1 - \theta_w^{-1}) \mathbb{E}_h\{\tilde{n}_t(h)\} \right)^2 \mathbb{V}_h\{\tilde{n}_t(h)\} + \mathcal{O}(||\zeta||^3).$$

Similarly

$$\mathbb{E}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\} = \mathbb{E}_h\left\{ \exp\left[ (1 - \theta_w^{-1}) \tilde{n}_t(h) \right] \right\}$$

so that, by applying lemma 1 once more, one obtains

$$\mathbb{E}_h\left\{ \left( \frac{N_t(h)}{N^n} \right)^{\frac{\theta_w - 1}{\theta_w}} \right\} = \exp\left[ (1 - \theta_w^{-1}) \mathbb{E}_h\{\tilde{n}_t(h)\} \right] \left( 1 + \frac{1}{2}(1 - \theta_w^{-1})^2 \mathbb{V}_h\{\tilde{n}_t(h)\} \right) + \mathcal{O}(||\xi||^3).$$

Then combining the previous results

$$\tilde{n}_t = \mathbb{E}_h\{\tilde{n}_t(h)\} + \frac{1}{2}\frac{1}{1 - \theta_w^{-1}} \frac{(1 - \theta_w^{-1})^2 \mathbb{V}_h\{\tilde{n}_t(h)\}}{\left( 1 + \frac{1}{2}(1 - \theta_w^{-1})^2 \mathbb{V}_h\{\tilde{n}_t(h)\} \right)^2} + \mathcal{O}(||\zeta||^3).$$

It is convenient to define

$$\Delta_{h,t} \equiv \mathbb{V}_h\{\tilde{n}_t(h)\}$$

so that finally

$$\tilde{n}_t = \mathbb{E}_h\{\tilde{n}_t(h)\} + Q_{0,h} + \frac{1-\theta_w^{-1}}{2} Q_{1,h}(\Delta_{h,t} - \Delta_n) + \mathcal{O}(||\zeta||^3).$$

where we defined

$$Q_{0,h} = \frac{\frac{1-\theta_w^{-1}}{2}\Delta_n}{\left[1+\frac{1}{2}(1-\theta_w^{-1})^2\Delta_n\right]^2}$$

and

$$Q_{1,h} = \frac{1-\frac{1}{2}(1-\theta_w^{-1})^2\Delta_n}{\left[1+\frac{1}{2}(1-\theta_w^{-1})^2\Delta_n\right]^3}$$

Applying the same logic on output and defining

$$\Delta_{y,t} \equiv \mathbb{V}_f\{\tilde{y}_t(f)\}$$

one gets

$$\tilde{y}_{z,t} = \mathbb{E}_f\{\tilde{y}_{z,t}(f)\} + Q_{0,y} + \frac{1-\theta_p^{-1}}{2} Q_{1,y}(\Delta_{y,t} - \Delta_y) + \mathcal{O}(||\zeta||^3).$$

where we defined

$$Q_{0,y} = \frac{\frac{1-\theta_p^{-1}}{2}\Delta_y}{\left[1+\frac{1}{2}(1-\theta_p^{-1})^2\Delta_y\right]^2}$$

and

$$Q_{1,y} = \frac{1-\frac{1}{2}(1-\theta_p^{-1})^2\Delta_y}{\left[1+\frac{1}{2}(1-\theta_p^{-1})^2\Delta_y\right]^3}$$

Then recall that

$$N_t = \int_0^1 L_t(f)\mathrm{d}f = \int_0^1 Y_{z,t}(f)^\phi \mathrm{d}f$$

which implies

$$\frac{N_t}{N^n} = \int_0^1 \left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^\phi \mathrm{d}f$$

where we used $N^n = (Y_z^n)^\phi$.

This relation rewrites

$$\tilde{n}_t = \log\left(\int_0^1 \left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^\phi \mathrm{d}f\right)$$

This expression is of the form

$$\tilde{n}_t = \log\left(\mathbb{E}_f\left\{\left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^\phi\right\}\right).$$

Using lemmas 1 and 2, we obtain the following three approximations

$$\tilde{n}_t = \mathbb{E}_f\{\phi(\tilde{y}_{z,t}(f) - z_t)\} + \frac{1}{2}\frac{\mathbb{V}_f\left\{\left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^\phi\right\}}{\left(\mathbb{E}_f\left\{\left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^\phi\right\}\right)^2} + \mathcal{O}(||\zeta||^3),$$

$$\mathbb{V}_f\left\{\left(\frac{Y_{z,t}(f)}{Y_z^n}\right)^{\phi}\right\} = \phi^2\left[\exp\left[\phi\mathbb{E}\{\tilde{y}_{z,t}(f)\}\right]\right]^2\mathbb{V}_f\{\tilde{y}_{z,t}(f)\} + \mathcal{O}(||\zeta||^3),$$

$$\mathbb{E}_f\left\{\left(\frac{y_{z,t}(f)}{y_z^n}\right)^{\phi}\right\} = \exp\left[\phi\mathbb{E}\{\tilde{y}_{z,t}(f)\}\right]\left(1+\frac{1}{2}\phi^2\mathbb{V}_f\{\tilde{y}_{z,t}(f)\}\right) + \mathcal{O}(||\zeta||^3).$$

Combining these expressions as before yields

$$\tilde{n}_t = \phi\mathbb{E}_f\{\tilde{y}_{z,t}(f)\} + \frac{1}{2}\phi^2\frac{\mathbb{V}_f\{\tilde{y}_{z,t}(f)\}}{\left(1+\frac{1}{2}\phi^2\mathbb{V}_f\{\tilde{y}_{z,t}(f)\}\right)^2} + \mathcal{O}(||\zeta||^3).$$

We finally obtain

$$\tilde{n}_t = \phi\mathbb{E}_f\{\tilde{y}_{z,t}(f)\} + P_{0,y} + \frac{1}{2}\phi^2 P_{1,y}(\Delta_{y,t}-\Delta_y) + \mathcal{O}(||\zeta||^3),$$

where we used

$$\frac{\mathbb{V}_f\{\tilde{y}_{z,t}(f)\}}{\left(1+\frac{1}{2}\phi^2\mathbb{V}_f\{\tilde{y}_{z,t}(f)\}\right)^2} = \frac{\Delta_y}{\left(1+\frac{1}{2}\phi^2\Delta_y\right)^2} + \frac{1-\frac{1}{2}\phi^2\Delta_y}{\left(1+\frac{1}{2}\phi^2\Delta_y\right)^3}(\Delta_{y,t}-\Delta_y) + \mathcal{O}(||\zeta||^3)$$

and defined

$$P_{0,y} = \frac{\frac{1}{2}\phi^2\Delta_y}{\left(1+\frac{1}{2}\phi^2\Delta_y\right)^2}$$

and

$$P_{1,y} = \frac{1-\frac{1}{2}\phi^2\Delta_y}{\left(1+\frac{1}{2}\phi^2\Delta_y\right)^3}$$

### L.3 Aggregate Price and Wage Levels

The aggregate price index is

$$P_t^{1-\theta_p} = \left(\int_0^1 P_t(f)^{1-\theta_p}\mathrm{d}f\right)$$

and the aggregate wage index is

$$W_t^{1-\theta_w} = \left(\int_0^1 W_t(h)^{1-\theta_w}\mathrm{d}h\right).$$

From lemma 1 and the definitions of $P_t$ and $W_t$, we obtain

$$p_t = \mathbb{E}_f\{p_t(f)\} + \frac{1}{2}\frac{1}{1-\theta_p}\frac{\mathbb{V}_f\{P_t(f)^{1-\theta_p}\}}{\mathbb{E}_f\{P_t(f)^{1-\theta_p}\}^2} + \mathcal{O}(||\zeta||^3),$$

and

$$w_t = \mathbb{E}_h\{w_t(h)\} + \frac{1}{2}\frac{1}{1-\theta_w}\frac{\mathbb{V}_h\{W_t(h)^{1-\theta_w}\}}{\mathbb{E}_h\{W_t(h)^{1-\theta_w}\}^2} + \mathcal{O}(||\zeta||^3).$$

Then, from lemma 2, we obtain

$$\begin{aligned}\mathbb{V}_f\{P_t(f)^{1-\theta_p}\} &= \mathbb{V}_f\{\exp[(1-\theta_p)p_t(f)]\}\\ &= (1-\theta_p)^2\exp[(1-\theta_p)\bar{p}_t]^2\Delta_{p,t} + \mathcal{O}(||\zeta||^3),\end{aligned}$$

and

$$\mathbb{V}_h\{W_t(h)^{1-\theta_w}\} = \mathbb{V}_h\{\exp[(1-\theta_w)w_t(h)]\}$$
$$= (1-\theta_w)^2\exp[(1-\theta_w)\bar{w}_t]^2\Delta_{w,t} + \mathcal{O}(||\zeta||^3),$$

where we defined

$$\bar{p}_t = \mathbb{E}_f\{p_t(f)\}, \quad \bar{w}_t = \mathbb{E}_h\{w_t(h)\},$$

$$\Delta_{p,t} = \mathbb{V}_f\{p_t(f)\}, \quad \Delta_{w,t} = \mathbb{V}_h\{w_t(h)\}.$$

Applying lemma 1 once again, we obtain

$$\mathbb{E}_f\{P_t(f)^{1-\theta_p}\} = \mathbb{E}_f\{\exp[(1-\theta_p)p_t(f)]\}$$
$$= \exp[(1-\theta_p)\bar{p}_t]\left(1+\frac{1}{2}(1-\theta_p)^2\Delta_{p,t}\right)$$

and

$$\mathbb{E}_h\{W_t(h)^{1-\theta_w}\} = \mathbb{E}_h\{\exp[(1-\theta_w)w_t(h)]\}$$
$$= \exp[(1-\theta_w)\bar{w}_t]\left(1+\frac{1}{2}(1-\theta_w)^2\Delta_{w,t}\right)$$

Combining these relations, we obtain

$$p_t = \bar{p}_t + \frac{1}{2}\frac{(1-\theta_p)\Delta_{p,t}}{\left[1+\frac{1}{2}(1-\theta_p)^2\Delta_{p,t}\right]^2} + \mathcal{O}(||\zeta||^3),$$

and

$$w_t = \bar{w}_t + \frac{1}{2}\frac{(1-\theta_w)\Delta_{w,t}}{\left[1+\frac{1}{2}(1-\theta_w)^2\Delta_{w,t}\right]^2} + \mathcal{O}(||\zeta||^3).$$

Thus

$$p_t = \bar{p}_t + Q_{0,p} + \frac{1-\theta_p}{2}Q_{1,p}(\Delta_{p,t}-\Delta_p) + \mathcal{O}(||\zeta||^3),$$

and

$$w_t = \bar{w}_t + Q_{0,w} + \frac{1-\theta_w}{2}Q_{1,w}(\Delta_{w,t}-\Delta_w) + \mathcal{O}(||\zeta||^3).$$

where we defined

$$Q_{0,p} = \frac{\frac{1-\theta_p}{2}\Delta_p}{\left[1+\frac{1}{2}(1-\theta_p)^2\Delta_p\right]^2}, \quad Q_{0,w} = \frac{\frac{1-\theta_w}{2}\Delta_w}{\left[1+\frac{1}{2}(1-\theta_w)^2\Delta_w\right]^2}$$

and

$$Q_{1,p} = \frac{1-\frac{1}{2}(1-\theta_p)^2\Delta_p}{\left[1+\frac{1}{2}(1-\theta_p)^2\Delta_p\right]^3}, \quad Q_{1,w} = \frac{1-\frac{1}{2}(1-\theta_w)^2\Delta_w}{\left[1+\frac{1}{2}(1-\theta_w)^2\Delta_w\right]^3}$$

Remark that the constant terms in the second-order approximation of the log-price index can be rewritten as

$$Q_{0,p} - \frac{1-\theta_p}{2} Q_{1,p}\Delta_p = \frac{1}{2}\frac{(1-\theta_p)^3\Delta_p^2}{\left[1+\frac{1}{2}(1-\theta_p)^2\Delta_p\right]^3}.$$

Finally, using the demand functions, one obtains

$$\tilde{y}_{z,t}(f) = -\theta_p[p_t(f) - p_t] + \tilde{y}_{z,t},$$

$$\tilde{n}_t(h) = -\theta_w[w_t(h) - w_t] + \tilde{n}_t,$$

from which we deduce that

$$\Delta_{y,t} = \theta_p^2\Delta_{p,t}$$

and

$$\Delta_{h,t} = \theta_w^2\Delta_{w,t}.$$

### L.4 Price and Wage Dispersions

We now derive the law of motion of price dispersion. Notice that

$$\Delta_{p,t} = \mathbb{V}_f\{p_t(f) - \bar{p}_{t-1}\}$$

Immediate manipulations of the definition of the cross-sectional mean of log-prices yield

$$\bar{p}_t - \bar{p}_{t-1} = \alpha_p\gamma_p\pi_{t-1} + (1-\alpha_p)[p_t^\star - \bar{p}_{t-1}]. \tag{L.1}$$

Then, the classic variance formula yields

$$\Delta_{p,t} = \mathbb{E}_f\{[p_t(f) - \bar{p}_{t-1}]^2\} - [\mathbb{E}_f\{p_t(f) - \bar{p}_{t-1}\}]^2$$

Using this, we obtain

$$\Delta_{p,t} = \alpha_p\mathbb{E}_f\{[p_{t-1}(f) - \bar{p}_{t-1} + \gamma_p\pi_{t-1}]^2\} + (1-\alpha_p)[p_t^\star - \bar{p}_{t-1}]^2 - [\bar{p}_t - \bar{p}_{t-1}]^2$$

Notice that

$$\begin{aligned}
&(1-\alpha_p)[p_t^\star - \bar{p}_{t-1}]^2 - [\bar{p}_t - \bar{p}_{t-1}]^2 \\
&= (1-\alpha_p)\left[\frac{1}{1-\alpha_p}(\bar{p}_t - \bar{p}_{t-1}) - \frac{\alpha_p}{1-\alpha_p}\gamma_p\pi_t)\right]^2 - [\bar{p}_t - \bar{p}_{t-1}]^2 \\
&= \frac{\alpha_p}{1-\alpha_p}[\bar{p}_t - \bar{p}_{t-1} - \gamma_p\pi_t]^2 - \alpha_p[\gamma_p\pi_t]^2
\end{aligned}$$

Using this in the above equation yields

$$\Delta_{p,t} = \alpha_p \mathbb{E}_f\{[p_{t-1}(f) - \bar{p}_{t-1} + \gamma_p \pi_t]^2\} - \alpha_p[\gamma_p \pi_t]^2 + \frac{\alpha_p}{1-\alpha_p}[\bar{p}_t - \bar{p}_{t-1} - \gamma_p \pi_t]^2$$

Now, notice also that

$$\alpha_p \mathbb{E}_f\{[p_{t-1}(f) - \bar{p}_{t-1}]^2\} = \alpha_p \mathbb{E}_f\{[p_{t-1}(f) - \bar{p}_{t-1} + \gamma_p \pi_t]^2\} - \alpha_p[\gamma_p \pi_t]^2$$

It then follows that

$$\Delta_{p,t} = \alpha_p \mathbb{E}_f\{[p_{t-1}(f) - \bar{p}_{t-1}]^2\} + \frac{\alpha_p}{1-\alpha_p}[\bar{p}_t - \bar{p}_{t-1} - \gamma_p \pi_t]^2$$

Hence

$$\Delta_{p,t} = \alpha_p \Delta_{p,t-1} + \frac{\alpha_p}{1-\alpha_p}[\bar{p}_t - \bar{p}_{t-1} - \gamma_p \pi_t]^2$$

Using

$$p_t = \bar{p}_t + Q_{0,p} + \frac{1-\theta_p}{2} Q_{1,p}(\Delta_{p,t} - \Delta_p) + \mathcal{O}(||\zeta||^3),$$

we obtain

$$\bar{p}_t - \bar{p}_{t-1} = \pi_t - \frac{1-\theta_p}{2} Q_{1,p}(\Delta_{p,t} - \Delta_{p,t-1}) + \mathcal{O}(||\zeta||^3).$$

Hence

$$\Delta_{p,t} = \alpha_p \Delta_{p,t-1} + \frac{\alpha_p}{1-\alpha_p}\left[\pi_t - \frac{1-\theta_p}{2} Q_{1,p}(\Delta_{p,t} - \Delta_{p,t-1}) - \gamma_p \pi_{t-1}\right]^2 + \mathcal{O}(||\zeta||^3).$$

The steady-state value of $\Delta_p$ is thus

$$\Delta_p = \frac{(1-\gamma_p)^2 \alpha_p}{(1-\alpha_p)^2}\pi^2$$

We obtain finally

$$\Delta_{p,t} = \alpha_p \Delta_{p,t-1} + \frac{\alpha_p}{1-\alpha_p}\left[(1-\gamma_p)\pi + \hat{\pi}_t - \gamma_p \hat{\pi}_{t-1} - \frac{1-\theta_p}{2} Q_{1,p}(\Delta_{p,t} - \Delta_{p,t-1})\right]^2 + \mathcal{O}(||\zeta||^3).$$

For sufficiently small $\pi$, price dispersion $\Delta_{p,t}$ is second-order.

We now derive the law of motion of wage dispersion. Following similar steps as for price dispersion, notice that

$$\Delta_{w,t} = \mathbb{V}_h\{w_t(h) - \bar{w}_{t-1}\}$$

Immediate manipulations of the definition of the cross-sectional mean of log-wages yield

$$\bar{w}_t - \bar{w}_{t-1} = \alpha_w(\gamma_z \mu_z + \gamma_w \pi_{t-1}) + (1-\alpha_w)[w_t^\star - \bar{w}_{t-1}]. \tag{L.2}$$

Then, the classic variance formula yields

$$\Delta_{w,t} = \mathbb{E}_h\{[w_t(h) - \bar{w}_{t-1}]^2\} - [\mathbb{E}_h\{w_t(h) - \bar{w}_{t-1}\}]^2$$

Using this, we obtain

$$\Delta_{w,t} = \alpha_w \mathbb{E}_h\{[w_{t-1}(h) - \bar{w}_{t-1} + \gamma_z\mu_z + \gamma_w\pi_{t-1}]^2\} + (1-\alpha_w)[w_t^\star - \bar{w}_{t-1}]^2 - [\bar{w}_t - \bar{w}_{t-1}]^2$$

Notice that

$$w_t^\star - \bar{w}_{t-1} = \frac{1}{1-\alpha_w}(\bar{w}_t - \bar{w}_{t-1}) - \frac{\alpha_w}{1-\alpha_w}[\gamma_z\mu_z + \gamma_w\pi_t]$$

so that

$$\begin{aligned}
&(1-\alpha_w)[w_t^\star - \bar{w}_{t-1}]^2 - [\bar{w}_t - \bar{w}_{t-1}]^2 \\
&= (1-\alpha_w)\left[\frac{1}{1-\alpha_w}(\bar{w}_t - \bar{w}_{t-1}) - \frac{\alpha_w}{1-\alpha_w}[\gamma_z\mu_z + \gamma_w\pi_t]\right]^2 - [\bar{w}_t - \bar{w}_{t-1}]^2 \\
&= \frac{\alpha_w}{1-\alpha_w}\left[\bar{w}_t - \bar{w}_{t-1} - [\gamma_z\mu_z + \gamma_w\pi_t]\right]^2 - \alpha_w[\gamma_z\mu_z + \gamma_w\pi_t]^2
\end{aligned}$$

Using this in the above equation yields

$$\begin{aligned}
\Delta_{w,t} = \alpha_w \mathbb{E}_h\{[w_{t-1}(h) - \bar{w}_{t-1} + \gamma_z\mu_z + \gamma_w\pi_t]^2\} \\
- \alpha_w[\gamma_w \log(1+\pi_t)]^2 + \frac{\alpha_w}{1-\alpha_w}\left[\bar{w}_t - \bar{w}_{t-1} - [\gamma_z\mu_z + \gamma_w\pi_t]\right]^2
\end{aligned}$$

Now, notice also that

$$\alpha_w \mathbb{E}_h\{[w_{t-1}(h) - \bar{w}_{t-1}]^2\} = \alpha_w \mathbb{E}_h\{[w_{t-1}(h) - \bar{w}_{t-1} + \gamma_z\mu_z + \gamma_w\pi_t]^2\} - \alpha_w[\gamma_z\mu_z + \gamma_w\pi_t]^2$$

It then follows that

$$\Delta_{w,t} = \alpha_w \mathbb{E}_h\{[w_{t-1}(h) - \bar{w}_{t-1}]^2\} + \frac{\alpha_w}{1-\alpha_w}\left[\bar{w}_t - \bar{w}_{t-1} - [\gamma_z\mu_z + \gamma_p\pi_t]\right]^2$$

Hence

$$\Delta_{w,t} = \alpha_w \Delta_{w,t-1} + \frac{\alpha_w}{1-\alpha_w}\left[\bar{w}_t - \bar{w}_{t-1} - [\gamma_z\mu_z + \gamma_w\pi_t]\right]^2$$

which, in turn, implies

$$\Delta_{w,t} = \alpha_w \Delta_{w,t-1} + \frac{\alpha_w}{1-\alpha_p w}\left[\bar{w}_t - \bar{w}_{t-1} - \gamma_z\mu_z - \gamma_w\pi_{t-1}\right]^2.$$

Using

$$w_t = \bar{w}_t + Q_{0,w} + \frac{1-\theta_w}{2}Q_{1,w}(\Delta_{w,t} - \Delta_w) + \mathcal{O}(||\zeta||^3),$$

we obtain

$$\bar{w}_t - \bar{w}_{t-1} = \pi_{w,t} - \frac{1-\theta_w}{2}Q_{1,w}(\Delta_{w,t} - \Delta_{w,t-1}) + \mathcal{O}(||\zeta||^3).$$

Hence

$$\begin{aligned}
\Delta_{w,t} = \alpha_w \Delta_{w,t-1} + \frac{\alpha_w}{1-\alpha_w}\Bigg[\pi_{w,t} - \frac{1-\theta_w}{2}Q_{1,w}(\Delta_{w,t} - \Delta_{w,t-1}) \\
- \gamma_z\mu_z - \gamma_w\pi_{t-1}\Bigg]^2 + \mathcal{O}(||\zeta||^3).
\end{aligned}$$

The steady-state value of $\Delta_w$ is thus

$$\Delta_w = \frac{\alpha_w}{(1-\alpha_w)^2}[(1-\gamma_z)\mu_z + (1-\gamma_w)\pi]^2$$

We obtain finally

$$\Delta_{w,t} = \alpha_w \Delta_{w,t-1} + \frac{\alpha_w}{1-\alpha_w}\Big[(1-\gamma_z)\mu_z + (1-\gamma_w)\pi + \hat{\pi}_{w,t} - \gamma_w \hat{\pi}_{t-1} \\ - \frac{1-\theta_w}{2} Q_{1,w}(\Delta_{w,t} - \Delta_{w,t-1})\Big]^2 + \mathcal{O}(||\zeta||^3).$$

For sufficiently small $\pi$ and $\mu_z$, wage dispersion $\Delta_{w,t}$ is second-order.

Because the steady-state value of $\Delta_p$ is of second-order, many of the expressions previously derived considerably simplify. In particular, we now obtain

$$p_t = \bar{p}_t + \frac{1-\theta_p}{2}\Delta_{p,t} + \mathcal{O}(||\zeta,\pi||^3),$$

$$w_t = \bar{w}_t + \frac{1-\theta_w}{2}\Delta_{w,t} + \mathcal{O}(||\zeta,\pi||^3),).$$

Now, because $\Delta_y$ and $\Delta_n$ are proportional to $\Delta_p$ and $\Delta_w$, respectively, and because $\Delta_p$ and $\Delta_w$ are both proportional to $\pi^2$, we also obtain

$$\tilde{n}_t = \mathbb{E}_h\{\tilde{n}_t(h)\} + \frac{1-\theta_w^{-1}}{2}\Delta_{h,t} + \mathcal{O}(||\zeta,\pi||^3),$$

$$\tilde{n}_t = \phi(\mathbb{E}_f\{\tilde{y}_{z,t}(f)\} - z_t) + \frac{1}{2}\phi^2\Delta_{y,t} + \mathcal{O}(||\zeta,\pi||^3),$$

$$\tilde{y}_t = \mathbb{E}_f\{\tilde{y}_t(f)\} + \frac{1-\theta_p^{-1}}{2}\Delta_{y,t} + \mathcal{O}(||\zeta,\pi||^3),).$$

Thus, for sufficiently small inflation rates, we obtain formulas resembling those derived in **?**.

Finally, price and wage dispersions rewrite

$$\Delta_{p,t} = \alpha_p\Delta_{p,t-1} + \frac{\alpha_p}{1-\alpha_p}\Big[(1-\gamma_p)\pi + \hat{\pi}_t - \gamma_p\hat{\pi}_{t-1}\Big]^2 + \mathcal{O}(||\zeta,\pi||^3),).$$

$$\Delta_{w,t} = \alpha_w\Delta_{w,t-1} + \frac{\alpha_w}{1-\alpha_w}\Big[(1-\gamma_z)\mu_z + (1-\gamma_w)\pi + \hat{\pi}_{w,t} - \gamma_w\hat{\pi}_{t-1}\Big]^2 + \mathcal{O}(||\zeta,\pi||^3),).$$

### L.5 Combining the Results

Combining the previous results, we obtain

$$\int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} dh = \chi(N^n)^{1+\nu} \Big[ \tilde{n}_t + \frac{1}{2}(1+\nu)\tilde{n}_t^2 + \tilde{n}_t \zeta_{h,t} + \frac{1}{2}(1+\nu\theta_w)\theta_w \Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

In turn, we have

$$\tilde{n}_t = \phi \tilde{y}_t + \frac{1}{2}\phi[(\phi-1)\theta_p + 1]\theta_p \Delta_{p,t} + \mathcal{O}(||\zeta, \pi||^3),$$

so that

$$\int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} dh = \phi\chi(N^n)^{1+\nu} \Big[ (\tilde{y}_t - z_t) + \frac{1}{2}(1+\nu)\phi\tilde{y}_t^2 + \tilde{y}_t \zeta_{h,t} + \frac{1}{2}[(\phi-1)\theta_p+1]\theta_p\Delta_{p,t} + \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w \Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

Then, using

$$(1-\Phi)\frac{1-\beta\eta}{1-\eta} = \phi\chi(N^n)^{1+\nu},$$

where we defined

$$1 - \Phi \equiv \frac{1+\tau_w}{\mu_w}\frac{1+\tau_p}{\mu_p},$$

we obtain

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} dh \right\} = (1-\Phi)\frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[ \tilde{y}_t + \frac{1}{2}(1+\nu)\phi\tilde{y}_t^2 + \tilde{y}_t \zeta_{h,t} + \frac{1}{2}[(\phi-1)\theta_p+1]\theta_p\Delta_{p,t} + \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w \Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

Assuming the distortions are themselves negligible, this simplifies further to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_0^1 \frac{\chi}{1+\nu} e^{\zeta_{h,t}} (N_t(h))^{1+\nu} dh \right\} = \frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[ (1-\Phi)\tilde{y}_t + \frac{1}{2}(1+\nu)\phi\tilde{y}_t^2 + \tilde{y}_t \zeta_{h,t} + \frac{1}{2}[(\phi-1)\theta_p+1]\theta_p\Delta_{p,t} + \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w \Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

We now deal with the first term in the utility function. To that end, notice that

$$\sum_{t=0}^{\infty} \beta^t a_{t-1} = a_{-1} + \beta \sum_{t=0}^{\infty} \beta^{t-1} a_{t-1} = a_{-1} + \beta \sum_{t=0}^{\infty} \beta^t a_t.$$

Using this trick, we obtain

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{\zeta_{c,t}} \log(C_{z,t} - \eta C_{z,t-1} e^{-\zeta_{z,t}}) = \frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[ \tilde{y}_{z,t} - \frac{1}{2}[\varphi(1+\beta\eta^2) - 1]\tilde{y}_{z,t}^2 + \eta\varphi\tilde{y}_{z,t}\tilde{y}_{z,t-1} + \varphi\hat{g}_t\tilde{y}_{z,t} - \eta\varphi\zeta_{z,t}^*\tilde{y}_{z,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta||^3),$$

where we defined

$$\varphi^{-1} \equiv (1-\beta\eta)(1-\eta),$$

$$\hat{g}_t = (1-\eta)(\zeta_{c,t} - \beta\eta\mathbb{E}_t\{\zeta_{c,t+1}\}),$$

so that

$$(1-\beta\eta)\varphi\hat{g}_t \equiv (\zeta_{c,t} - \beta\eta\mathbb{E}_t\{\zeta_{c,t+1}\}).$$

and

$$\zeta_{z,t}^* = \zeta_{z,t} - \beta\mathbb{E}_t\{\zeta_{z,t+1}\}$$

Combining terms, we obtain

$$\mathsf{U}_0 = \frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[ \Phi\tilde{y}_{z,t} - \frac{1}{2}[\varphi(1+\beta\eta^2) + \omega]\tilde{y}_{z,t}^2 + \eta\varphi\tilde{y}_{z,t}\tilde{y}_{z,t-1} + (\varphi\hat{g}_t - \zeta_{h,t} - \varphi\eta\zeta_{z,t}^*)\tilde{y}_{z,t} - \frac{1}{2}[(\phi-1)\theta_p + 1]\theta_p\Delta_{p,t} - \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w\Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta||^3),$$

where, as defined earlier

$$\omega = (1+\nu)\phi - 1$$

Now, recall that

$$[\varphi(1+\beta\eta^2) + \omega]\hat{y}_{z,t}^n - \varphi\beta\eta\mathbb{E}_t\left\{\hat{y}_{z,t+1}^n\right\} - \varphi\eta\hat{y}_{z,t-1}^n = \varphi\hat{g}_t - \zeta_{h,t} - \varphi\eta\zeta_{z,t}^*$$

Using this above yields

$$\mathsf{U}_0 = \frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[ \Phi\tilde{y}_{z,t} - \frac{1}{2}[\varphi(1+\beta\eta^2) + \omega]\tilde{y}_{z,t}^2 + \eta\varphi\tilde{y}_{z,t}\tilde{y}_{z,t-1} + [\varphi(1+\beta\eta^2) + \omega]\hat{y}_{z,t}^n\tilde{y}_{z,t} - \varphi\beta\eta\hat{y}_{z,t+1}^n\tilde{y}_{z,t} - \varphi\eta\hat{y}_{z,t-1}^n\tilde{y}_{z,t} - \frac{1}{2}[(\phi-1)\theta_p + 1]\theta_p\Delta_{p,t} - \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w\Delta_{w,t} \Big] + \text{t.i.p} + \mathcal{O}(||\zeta||^3),$$

$$\mathsf{U}_0 = \frac{1-\beta\eta}{1-\eta}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg[\Phi\tilde{y}_t - \frac{1}{2}[\varphi(1+\beta\eta^2)+\omega]\tilde{y}_t^2 + \eta\varphi\tilde{y}_t\tilde{y}_{t-1}$$

$$+ [\omega+\varphi(1+\beta\eta^2)]\hat{y}_t^n\tilde{y}_t - \varphi\beta\eta\hat{y}_{t+1}^n\tilde{y}_t - \varphi\eta\hat{y}_{t-1}^n\tilde{y}_t$$

$$-\frac{1}{2}[(\phi-1)\theta_p+1]\theta_p\Delta_{p,t} - \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w\Delta_{w,t}\Bigg] + \text{t.i.p} + \mathcal{O}(||\xi,\pi||^3)$$

To simplify this expression, we seek constant terms $\delta_0$, $\delta$ and $x^\star$ such that

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg\{-\frac{1}{2}\delta_0[(\tilde{y}_t-\hat{y}_t^n)-\delta(\tilde{y}_{t-1}-\hat{y}_{t-1}^n)-x^\star]^2\Bigg\}$$

$$= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg[\Phi\tilde{y}_t - \frac{1}{2}[\varphi(1+\beta\eta^2)+\omega]\tilde{y}_t^2 + \eta\varphi\tilde{y}_t\tilde{y}_{t-1}$$

$$+ [\omega+\varphi(1+\beta\eta^2)]\tilde{y}_t\hat{y}_t^n - \varphi\beta\eta\tilde{y}_t\hat{y}_{t+1}^n - \varphi\eta\tilde{y}_t\hat{y}_{t-1}^n\Bigg] + \text{t.i.p}$$

Developing yields

$$-\frac{\delta_0}{2}\Big[(\tilde{y}_t-\hat{y}_t^n)-\delta(\tilde{y}_{t-1}-\hat{y}_{t-1}^n)-x^*\Big]^2$$

$$= -\frac{1}{2}\delta_0\tilde{y}_t^2 + \delta_0\tilde{y}_t\hat{y}_t^n + \delta_0\delta\tilde{y}_t\tilde{y}_{t-1} - \delta_0\delta\tilde{y}_t\hat{y}_{t-1}^n - \delta_0\delta\tilde{y}_{t-1}\hat{y}_t^n$$

$$-\frac{1}{2}\delta_0\delta^2\tilde{y}_{t-1}^2 + \delta_0\delta^2\tilde{y}_{t-1}\hat{y}_{t-1}^n + \delta_0(\tilde{y}_t-\delta\tilde{y}_{t-1})\hat{x}^* + \text{t.i.p}$$

Thus

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg\{-\frac{\delta_0}{2}[(\tilde{y}_t-\hat{y}_t^n)-\delta(\tilde{y}_{t-1}-\hat{y}_{t-1}^n)-x^*]^2\Bigg\}$$

$$= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg\{\delta_0(1-\beta\delta)\hat{x}^*\tilde{y}_t - \frac{1}{2}\delta_0(1+\beta\delta^2)\tilde{y}_t^2 + \delta_0\delta\tilde{y}_t\tilde{y}_{t-1}$$

$$+ \delta_0(1+\beta\delta^2)\tilde{y}_t\hat{y}_t^n - \delta_0\delta\tilde{y}_t\hat{y}_{t-1}^n - \delta_0\delta\beta\tilde{y}_t\hat{y}_{t+1}^n\Bigg\} + \text{t.i.p}$$

Identifying term by term, we obtain

$$\delta_0(1-\beta\delta)x^* = \Phi,$$

$$\delta_0(1+\beta\delta^2) = [\omega+\varphi(1+\beta\eta^2)],$$

$$\delta_0\delta = \eta\varphi,$$

Recall that the steady-state subsidy rates $\tau_p$ and $\tau_w$ are chosen to neutralize markups. Then, it follows that $\Phi = x^* = 0$.

Combining these relations, we obtain

$$\eta\delta^2 - \frac{\omega + \varphi)(1+\beta\eta^2)}{\beta\varphi}\delta + \eta\beta^{-1} = 0,$$

or equivalently

$$\mathbb{P}(\varkappa) = \beta^{-1}\varkappa^2 - \chi\varkappa + \eta^2 = 0,$$

where

$$\varkappa = \frac{\eta}{\delta},$$

$$\chi = \frac{\omega + \varphi(1+\beta\eta^2)}{\beta\varphi} > 0.$$

Notice that

$$\mathbb{P}(0) = \eta^2 > 0,$$

$$\mathbb{P}(1) = -\frac{\omega}{\beta\varphi} < 0$$

so that the two roots of $\mathbb{P}(\varkappa) = 0$ obey

$$0 < \varkappa_1 < 1 < \varkappa_2.$$

In the sequel, we focus on the larger root and define

$$\varkappa = \varkappa_2 = \frac{\beta}{2}\left(\chi + \sqrt{\chi^2 - 4\eta^2\beta^{-1}}\right) > 1.$$

Since $\delta = \eta/\varkappa$, we have

$$0 \leq \delta \leq \eta < 1.$$

Thus, given the obtained value for $\varkappa$, we can deduce $\delta$ from which we can compute $\delta_0$.

We thus obtain

$$\mathsf{U}_0 = -\frac{1-\beta\eta}{1-\eta}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{\delta_0}{2}[(\tilde{y}_t - \hat{y}_t^n) - \delta(\tilde{y}_{t-1} - \hat{y}_{t-1}^n) - x^*]^2$$

$$+\frac{1}{2}[(\phi-1)\theta_p + 1]\theta_p\Delta_{p,t} + \frac{1}{2}(1+\nu\theta_w)\phi^{-1}\theta_w\Delta_{w,t}\Bigg\} + \text{t.i.p} + \mathcal{O}(||\zeta,\pi||^3),)$$

The last step consists in expressing price and wage dispersions in terms of squared price and wage inflations.

Recall that

$$\Delta_{p,t} = \alpha_p\Delta_{p,t-1} + \frac{\alpha_p}{1-\alpha_p}\Big[(1-\gamma_p)\pi + \hat{\pi}_t - \gamma_p\hat{\pi}_{t-1}\Big]^2 + \mathcal{O}(||\zeta,\pi||^3),).$$

Iterating backward on this formula yields

$$\Delta_{p,t} = \frac{\alpha_p}{1-\alpha_p} \sum_{s=0}^{t} \alpha_p^{t-s} [(1-\gamma_p)\pi + \hat{\pi}_s - \gamma_p \hat{\pi}_{s-1}]^2 + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

It follows that

$$\sum_{t=0}^{\infty} \beta^t \Delta_{p,t} = \frac{\alpha_p}{(1-\alpha_p)(1-\beta\alpha_p)} \sum_{t=0}^{\infty} \beta^t [(1-\gamma_p)\pi + \hat{\pi}_t - \gamma_p \hat{\pi}_{t-1}]^2 + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

and by the same line of reasoning

$$\sum_{t=0}^{\infty} \beta^t \Delta_{w,t} = \frac{\alpha_w}{(1-\alpha_w)(1-\beta\alpha_w)} \sum_{t=0}^{\infty} \beta^t [(1-\gamma_z)\mu_z + (1-\gamma_w)\pi + \hat{\pi}_{w,t} - \gamma_w \hat{\pi}_{t-1}]^2 + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3), ).$$

Thus, defining

$$\lambda_y \equiv \delta_0$$

$$\lambda_p \equiv \frac{\alpha_p \theta_p [(\phi - 1)\theta_p + 1]}{(1-\alpha_p)(1-\beta\alpha_p)}$$

$$\lambda_w \equiv \frac{\alpha_w \phi^{-1} \theta_w (1 + \nu\theta_w)}{(1-\alpha_w)(1-\beta\alpha_w)}$$

Using this and recalling that $x^* = 0$, the second order approximations to welfare rewrites

$$\mathsf{U}_0 = -\frac{1}{2}\frac{1-\beta\eta}{1-\eta} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \lambda_y [\hat{x}_t - \delta \hat{x}_{t-1} + (1-\delta)\bar{x}]^2 + \lambda_p [(1-\gamma_p)\pi + \hat{\pi}_t - \gamma_p \hat{\pi}_{t-1}]^2$$

$$+ \lambda_w [(1-\gamma_z)\mu_z + (1-\gamma_w)\pi + \hat{\pi}_{w,t} - \gamma_w \hat{\pi}_{t-1}]^2 \Big\} + \text{t.i.p} + \mathcal{O}(||\zeta, \pi||^3),$$

where we defined

$$\hat{x}_t \equiv \hat{y}_t - \hat{y}_t^n$$

$$\bar{x} \equiv \log\left(\frac{Y_z}{Y_z^n}\right).$$