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Multiple equilibria in a growth model with monopolistic competition*

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Summary. The standard neoclassical growth model is modified by introducing a market structure characterized by monopolistic competition and variable demand elasticities. In equilibrium, the price elasticity of the demand schedule facing a typical firm is a function of the aggregate savings rate. The latter feature results from an assumed wedge between the elasticity of substitution across goods in productive activities and that in consumption. In contrast with most examples in the literature our model does not require increasing returns (internal or external) in order to generate multiple equilibria.

1 Introduction

The neoclassical model, as developed by Cass (1965) and Koopmans (1965), remains the cornerstone of modern growth theory. Much of its success is, undoubtedly, tied to the sharpness of its predictions: given an initial capital stock the competitive equilibrium is unique, it corresponds to the optimal dynamic allocation, and is characterized by a monotonic convergence to a unique steady state[1]. Given identical preferences and technologies, two economies that start out with different per capita output levels should converge over time in terms of that variable.

Much of the research in growth theory since Cass-Koopmans has explored the implications of relaxing some of the assumptions underlying their model. Thus, researchers have analyzed models with more than one sector,[2] nondecreasing returns to capital,[3] overlapping generations of finite-lived agents,[4] and models with

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[1] Or a unique "balanced growth path," in the presence of exogenous technical progress.

[2] See Boldrin and Woodford (1990), Section 3, for a recent survey.

[3] See, e.g., Romer (1986), Rebelo (1991), and Jones and Manuelli (1990).

[4] See, e.g., Galor and Ryder (1989) and Galor (1992).

nonseparable preferences,[5] as well as models that combine several of the above features.

In the present paper I examine the equilibrium consequences of a variation on the Cass-Koopmans framework which consists of introducing a market structure characterized by monopolistic competition and variable demand elasticities. The potential role of variable demand elasticities in affecting the dynamics of capital accumulation can be illustrated by looking at the expression for the equilibrium interest rate R with results from profit maximization by a set of (symmetric) monopolistic competitors:

$$R = \left(1 - \frac{1}{\xi}\right) f'(k) - \delta \tag{1.1}$$

where k is the (suitably normalized) aggregate capital stock, $f'(k)$ is the marginal product of capital, ξ is the price elasticity of the demand schedule facing a typical firm, and δ is the rate of depreciation. (1.1) implies that, for any given k, the interest rate is increasing in the demand elasticity. The intuition behind that result is straightforward: when considering whether to employ or not an additional unit of capital, each firm recognizes that it will have to lower the price of its good if the market is to absorb the resulting marginal product. The implied lower return reduces the rental price the firm is willing to pay for the additional capital and, thus, the return accruing to capital owners (i.e., the interest rate). The lower the price elasticity, the greater the price reduction required and, ceteris paribus, the lower the interest rate. Since savings decisions – and, thus, the rate at which capital is accumulated – are influenced by current and anticipated future returns, it follows that changes in demand conditions will affect the dynamics of capital accumulation and growth.

The structure of the model presented below is such that, under certain assumptions, a positive relation between the demand elasticity and the aggregate savings rate arises in equilibrium. In that case, expectations of high current and future aggregate savings tend to increase private returns to investment, offsetting the negative impact of the associated declining marginal product of capital. As shown below, if the "complementarity" between aggregate and private savings decisions is sufficiently strong multiple steady states and multiple perfect foresight equilibrium paths may exist.[6]

In the model, the crucial dependence of the demand elasticity on aggregate savings arises, in equilibrium, from the postulated presence of a wedge between (a) the elasticity of substitution across inputs in production, and (b) the elasticity of substitution across goods in consumption. The typical firm sells its good to both consumers and producers and, by assumption, it is unable to price discriminate between those customer types. As a result, the effective price elasticity of its demand schedule depends on the composition of its demand. In a symmetric equilibrium the

[5] See, e.g., Heal and Ryder (1973).

[6] In Galí (1994a) I used a similar framework (augmented with leisure choice) to show the possibility of sunspot fluctuations in a neighborhood of a single steady state.

latter is a function of the aggregate savings rate. Whenever (a) is greater than (b) the positive relation necessary for multiple equilibria discussed above obtains.

A word is due about the related literature. Several recent papers have also introduced elements of imperfect competition in the context of growth models. In many of those models market power provides a simple device to sustain nonconvex technologies in equilibrium, without playing a direct role as a source of any of the interesting features of the resulting equilibrium dynamics (which are driven by the nonconvexities).[7] In models with horizontal innovation, the interaction between market power, dynamic increasing returns, and the nature of technology generates pecuniary externalities that often lead to multiple equilibria.[8] Also, in the work of some authors, the prospects of temporary monopoly rents plays a key role in eliciting research effort or implementation of innovations by a firm; the fact that the size of those rents depends, in turn, on the research or implementation activities of other firms often leads to nontrivial equilibrium dynamics.[9] All the previous models, however, rely on some form of increasing returns, and/or are characterized by constant markups. Furthermore, many of those models differ from the standard neoclassical growth model in several other dimensions (e.g., absence of capital accumulation). In contrast, the model below is constructed so that the presence of nontrivial, variable markups is the only departure from the Cass-Koopmans framework and, thus, the only possible source of its differential equilibrium dynamics.[10] In fact, the Cass-Koopmans equilibrium corresponds to a limiting case of the model below.

The paper is organized as follows. Section 2 lays out the basic model and notation. Section 3 derives the conditions characterizing an equilibrium. Section 4 analyzes the conditions for multiplicity stationary equilibria and their (local) stability properties. Section 5 analyzes the model's equilibrium dynamics. Section 6 briefly touches on some of its empirical and policy implications.

2 The model

2.1 Consumers

An infinite lived representative consumer seeks to maximize

$$\left(\frac{1}{1-\gamma}\right)\int_0^\infty c(t)^{1-\gamma}\exp(-\rho t)dt$$

[7] In fact the need for market power is sometimes avoided by assuming that increasing returns are external to the firm; see Benhabib and Farmer (1992) for a discussion of the equivalence between both structures and Boldrin and Rustichini (1994) for a careful analysis of the conditions for uniqueness of equilibria in models with external increasing returns.

[8] See, e.g., Ciccone and Matsuyama (1992) and Young (1993). We use the term "dynamic increasing returns" to refer to technologies characterized by a start-up cost (e.g., the price of a patent or an entry cost), with subsequent production carried out under constant returns.

[9] See, e.g., Judd (1985), Shleifer (1986), and Aghion and Howitt (1992).

[10] Some other recent examples of growth models in which the market power is the fundamental source of multiple equilibria include Zilibotti (1994), Galí (1995), and Galí and Zilibotti (1995). The previous models are characterized by markup variations resulting from entry and exit in imperfectly competitive industries

subject to the dynamic budget constraint and terminal conditions

$$\dot{a}(t) = w(t) - e(t) + \int_0^\infty (d_j(t) + \dot{q}_j(t))s_j(t)dj \tag{2.1}$$

$$\lim_{T\to\infty} a(T) \geq 0; \quad s_j(0) = 1, \textit{all } j \in [0,1] \tag{2.2}$$

where $c(t) \equiv (\int_0^1 c_j(t)^{\sigma - 1/\sigma} dj)^{\sigma/\sigma - 1}$ is a CES composite consumption index with elasticity of substitution $\sigma > 1$. $c_j(t)$ is the consumption flow of good j, which can be purchased at a unit price $p_j(t)$, for $j \in [0,1]$. $e(t) \equiv \int_0^1 p_j(t)c_j(t)dj$ denotes the flow of expenditure on consumption goods. Notice that we assume the existence of a continuum of differentiated goods, represented by the unit interval. Each good is produced by a different firm. $s_j(t)$ denotes the number of shares in firm j held at time t by the consumer. A share in firm j trades at price $q_j(t)$ and generates a dividend flow $d_j(t)$ at time t. Financial wealth is thus given by $a(t) \equiv \int_0^1 q_j(t)s_j(t)dj$. $w(t)$ denotes the wage flow accruing to the consumer, in exchange for the (inelastic) supply of one unit of labor services.[11]

The problem above can be solved in two stages. In a first stage the consumer decides how to allocate a given expenditure flow e among the different goods. This yields the system of demand equations (Dixit and Stiglitz (1977)):

$$c_j(t) = \left(\frac{p_j(t)}{P(t)}\right)^{-\sigma}\left(\frac{e(t)}{P(t)}\right) \tag{2.3}$$

for all $j \in [0,1]$, where $P(t) \equiv (\int_0^1 p(t)^{1-\sigma})^{1/1-\sigma}$. Furthermore, plugging (2.3) in the definition of $e(t)$ we obtain $c(t) = \frac{e(t)}{P(t)}$.

The second stage of the consumer problem consists of choosing the path of expenditures that maximizes $\left(\frac{1}{1-\gamma}\right)\int_0^\infty \left(\frac{e(t)}{P(t)}\right)^{1-\gamma} \exp(-\rho t)dt$ subject to (2.1) and (2.2). Appendix 1 derives the optimality conditions for this problem using the maximum principle. As shown in the appendix, those conditions can be combined to yield the familiar stock pricing equation

$$\frac{q_j(t)}{P(t)} = \int_0^\infty v_t(s)\left(\frac{d_j(t+s)}{P(t+s)}\right)ds \tag{2.4}$$

for all $j \in [0,1]$, where $v_t(s) \equiv \left(\frac{c(t+s)}{c(t)}\right)^{-\gamma} \exp(-\rho s)$ corresponds to the marginal rate of substitution between time t and time $t+s$ consumption.

[11] In Galí (1994b) I allowed for a constant growth rate in the quantity of labor services supplied (in terms of efficiency units). As in the neoclassical model that assumption introduces a source of exogenous long term growth though, after a suitable transformation of its variables, the associated equilibrium dynamics are (essentially) unchanged. In order to save space and notation we have chosen to ignore growth here.

2.2 Firms

There is a continuum of monopolistically competitive firms indexed by $j \in [0, 1]$. At each point in time a typical firm (say, firm j) uses the services of capital $k_j(t)$ and labor $l_j(t)$ to produce a quantity $y_j(t)$ of a differentiated good with a production function

$$y_j(t) = F(k_j(t), l_j(t)) \tag{2.5}$$

where F is assumed to be homogenous of degree one, twice differentiable, strictly concave, and strictly increasing in both arguments. We also assume the Inada conditions $\lim_{x \to 0} f'(x) = +\infty$ and $\lim_{x \to \infty} f'(x) = 0$, where $f(x) \equiv F(x, 1)$ represents the production function in intensive form.

Let i_{jh} denote the flow of purchases by firm j of the good produced by firm h. It is assumed that firm j's (gross) investment, denoted by i_j, is given at each point time by a CES function of such quantities.[12] Formally, $i_j(t) \equiv (\int_0^1 i_{jh}(t)^{\eta - 1/\eta} dh)^{\eta/\eta - 1}$, where $\eta > 1$ denotes the elasticity of substitution between different goods from the view-point of the firm which uses them as inputs. The firm's capital stock thus evolves according to the differential equation

$$\dot{k}_j(t) = i_j(t) - \delta k_j(t)$$

Thus we see that each firm sells the goods it produces to two customer types: (a) consumers, who derive utility from its consumption, and (b) other firms, which use it to increase their capital stock. We assume that firms cannot price discriminate between those two markets, thus ruling out obvious arbitrage opportunities.

At each point in time, and conditional on its level of investment expenditures $z_j(t) \equiv \int_0^1 p_h(t) i_{jh}(t) dh$, it is optimal for firm j to maximize its gross investment $i_j(t)$. That yields a set of investment demand functions

$$i_{jh}(t) = \left(\frac{p_h(t)}{\Pi(t)}\right)^{-\eta} \left(\frac{z_j(t)}{\Pi(t)}\right) \tag{2.6}$$

for all $j \in [0, 1]$, where $\Pi(t) \equiv (\int_0^1 p_h(t)^{1-\eta} dh)^{1/1-\eta}$. Furthermore, plugging (2.6) in the definition of $z_j(t)$ we obtain $i_j(t) = \left(\frac{z_j(t)}{\Pi(t)}\right)$.

The total demand for the good produced by a typical firm will be the sum of the demands coming from the two markets: consumers and firms:

$$y_j(t) = \left(\frac{p_j(t)}{P(t)}\right)^{-\sigma} \left(\frac{e(t)}{P(t)}\right) + \left(\frac{p_j(t)}{\Pi(t)}\right)^{-\eta} \left(\frac{z(t)}{\Pi(t)}\right) \tag{2.7}$$

where $z(t) \equiv \int_0^1 z_h(t) dh$.

In general the elasticity of substitution in consumption σ may be different from that in production η, since both activities are of different nature. As a result, the effective price elasticity of the demand for good j (denoted by ξ_j) will vary with the

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[12] This specification was first introduced by Kiyotaki (1988). None of the results would be affected if instead we introduced a perfectly competitive capital goods sector, producing a single capital good with the CES production function with elasticity parameter η

composition of that demand according to

$$\xi_j(t) = \xi(\lambda_j(t)) \equiv \lambda_j(t)\eta + (1 - \lambda_j(t))\sigma$$

where $\lambda_j(t) \equiv \left(\frac{1}{y_j(t)}\right)\left(\frac{p_j(t)}{\Pi(t)}\right)^{-\eta}\left(\frac{z(t)}{\Pi(t)}\right)$ represents the weight of the investment component (i.e., other firms purchases) in the demand for good j.

Given (2.4), maximization of wealth of its shareholds requires that each firm solves the following infinite horizon optimization problem (as of time 0, and letting $v(t) \equiv v_0(t)$):

$$\max \int_0^\infty v(t)\left(\frac{d_j(t)}{P(t)}\right)dt$$

subject to

$$d_j(t) = p_j(t)y_j(t) - w(t)l_j(t) - z_j(t) \tag{2.8}$$

$$\dot{k}_j(t) = \left(\frac{z_j(t)}{\Pi(t)}\right) - \delta k_j(t) \tag{2.9}$$

$$z_j(t) \geq 0; \quad k_j(t) \geq 0 \tag{2.10}$$

together with (2.5), (2.7), and the initial condition $k_j(0) = k(0)$. The solution to that optimal control problem faced by the firm can be characterized using the maximum principle. Let $\mu(x) \equiv \frac{\xi(x)}{\xi(x) - 1}$, defined for $x \in [0, 1]$. The optimal path of firm j's variables must satisfy:

$$\frac{p_j(t)F_2(k_j(t), l_j(t))}{\mu(\lambda_j(t))} = w(t) \tag{2.11}$$

$$\frac{p_j(t)F_1(k_j(t), l_j(t))}{\mu(\lambda_j(t))} - P(t)\delta\theta(t) + P(t)\dot{\theta}(t) = -\left(\frac{\dot{v}(t)}{v(t)}\right)P(t)\theta(t) \tag{2.12}$$

$$\theta(t) \leq \frac{\Pi(t)}{P(t)} \tag{2.13}$$

where $\theta(t)$ is the multiplier associated with (2.9), and (2.13) holds as a strict inequality only if $z_j(t) = 0$.[13] Notice that the presence of μ in (2.11) and (2.12) adjusts the marginal product of both capital and labor, reflecting the fact that the firm's recognition (and optimal use) of its market power. Both the demand for labor and capital (and, as a result, equilibrium wages and interest rates) are affected by this deviation from price-taking behavior. As shown below that effect has a potentially

[13] The necessity of (2.11)–(2.13) follows from proposition 7 in Arrow and Kurz (1970). Theorem 3 in Araujo and Scheinkman (1983) can be invoked to guarantee that, together with the transversality condition $\text{limit}_{T\to\infty} k_j(T)v(T)\theta(T) = 0$, are jointly necessary and sufficient for a maximum.

important impact on consumer's savings and the dynamics of capital accumulation.[14]

3 Equilibrium

Given the symmetry embedded in the model, all firms will set the same price and produce the same quantity. Accordingly, $p_j(t) = P(t) = \Pi(t)$, $c_j(t) = c(t)$, $i_{jh}(t) = i_j(t) \equiv i(t)$, $k_j(t) = k(t)$, $l_j(t) = l(t)$, $y_j(t) = y(t)$, for all h, $j \in [0, 1]$, and all t. Equilibrium in the goods and labor markets implies

$$c(t) + i(t) = f(k(t)) \tag{3.1}$$

all t.[15] The share of firms' purchases in the demand for each good, λ_j, will also be the same across firms, and will correspond to the aggregate savings rate:

$$\lambda(t) = 1 - \frac{c(t)}{f(k(t))} \tag{3.2}$$

Using (2.12), (3.1), (3.2), and the definition of v, we can characterize the equilibrium of the economy as a dynamical system in (k, c). Given any initial capital stock k_0, any trajectory satisfying

$$\dot{c} = \left(\frac{c}{\gamma}\right)\left(\frac{f'(k)}{\mu\left(1 - \frac{c}{f(k)}\right)} - (\delta + \rho)\right) \tag{3.3}$$

$$\dot{k} = f(k) - \delta k - c \tag{3.4}$$

$$c \geq 0, k \geq 0, f(k) - c \geq 0 \tag{3.5}$$

$$k(0) = k_0 \tag{3.6}$$

$$\lim_{T \to \infty} c(T)^{-\gamma} k(T) \exp(-\rho T) = 0 \tag{3.7}$$

constitutes an (interior) equilibrium of our model economy.[16] (hereafter the dependency of each variable on time is no longer made explicit, unless necessary, in order to ease the notation).

Given an equilibrium trajectory for k and c it is easy to determine the equilibrium values for the remaining variables, including investment ($i = f(k) - c$), output ($y = f(k)$), the savings rate $\left(\lambda = 1 - \frac{c}{f(k)}\right)$, the real wage $\left(\frac{w}{P} = \frac{f(k) - kf'(k)}{\mu(\lambda)}\right)$, real dividends $\left(\frac{d}{P} = c - \frac{w}{P}\right)$, and the price level $\left(P(t) = \left(\frac{c(t)}{c(0)}\right)^{-\gamma} \exp(-\rho t)\right)$.

[14] In contrast, and given the assumption of an inelastic labor supply, employment will not be affected by the existence of market power in the goods market.

[15] Notice that labor market equilibrium implies $l(t) = 1$, all t.

[16] When $z(t) \geq 0$ is binding, the equilibrium dynamics are trivially given by $c(t) = f(k(t))$ and $\dot{k}(t) = -\delta k(t)$.

If we set $\mu \equiv 1$, equations (3.3)–(3.7) characterize the optimal growth path of the neoclassical growth model or, equivalently, the associated equilibrium path under perfect competition. The Cass-Koopmans model thus emerges as a particular case of the model above. On the other hand, as long as both σ and η are finite and firms recognize (and exercise) their market power, μ will be greater than one and the economy's equilibrium allocation will differ from the perfectly competitive (and efficient) one. As it turns out, whether the equilibrium dynamics under imperfect competition are qualitatively similar or not to those characterizing the perfectly competitive economy depends on several features of technology and preferences. The remainder of the paper is devoted to the analysis and characterization of those dynamics.

4 Analysis of stationary equilibria

Let $\bar{k}$ denote the maximum sustainable capital stock, defined by $f(\bar{k}) - \delta\bar{k} = 0$. Henceforth we confine ourselves to equilibria in the interval $[0, \bar{k}]$. Define

$$R(k) \equiv \frac{f'(k)}{\mu\left(\frac{\delta k}{f(k)}\right)} - \delta \tag{4.1}$$

which corresponds to the equilibrium interest rate when savings exactly compensate capital depreciation, i.e., $\lambda = \frac{\delta k}{f(k)}$. Given (3.3)–(3.4), any (interior) stationary level of the captial stock, denoted by k^*, must satisfy

$$R(k^*) = \rho \tag{4.2}$$

The assumed properties of f and μ guarantee that R is continuously differentiable on $(0, \bar{k})$, satisfying

$$\lim_{x \to \bar{k}} R(x) = f'(\bar{k})\left(1 - \frac{1}{\eta}\right) - \delta < \rho \tag{4.3}$$

$$\lim_{x \to 0} R(x) = +\infty \tag{4.4}$$

The above properties guarantee the existence of at least one solution to (4.2), i.e., the existence of a stationary equilibrium.[17] Under what conditions will multiple stationary equilibria, i.e., multiple solutions to (4.2) exist? Given the properties of R, a necessary and sufficient condition for multiple stationary equilibria is the existence of at least one solution k^* of (4.2) such that $R'(k^*) > 0$. Figure 1 illustrates the previous statement, whose straightforward proof we omit.

Given the concavity of f, it is clear that $\mu' < 0$ is a necessary (though not sufficient) condition for multiple stationary equilibria. That condition is satisfied if and only if $\eta > \sigma$, i.e., whenever the elasticity of substitution among goods in

[17] It can be easily checked that, as long as σ and η are finite any solution to (4.2) lies below the "modified golden rule" capital stock, i.e., the stationary capital stock associated with the efficient allocation.

production is greater than in consumption. In that case R will be increasing over some range of k as long as (a) the marginal product of capital does not diminish too rapidly, (b) the average product of capital decreases sufficiently fast, and (c) η is sufficiently greater than σ. Under those conditions the possibility of multiple stationary equilibria emerges.

5 Equilibrium dynamics

The analysis of equilibrium paths can be conducted using the phase diagram associated with the system of differential equations (3.3)–(3.4). As in the neoclassical growth model the $\dot{k}=0$ locus takes the form of a hump-shaped curve on the (k,c) plane, given by the function $c=f(k)-\delta k$, defined on $[0,\bar{k}]$. The generic shape of that locus can be found in Figures 2 and 3. Trajectories above (below) the $\dot{k}=0$ locus are associated with decreasing (increasing) levels of capital. Notice that the previous locus always lies below the curve $c=f(k)$ (also shown in the figures), and which represents an upper bound for the level of consumption.

The $\dot{c}=0$ locus is implicitly given by the equation $\dfrac{f'(k)}{\mu(1-c/f(k))}=\delta+\rho$ (in addition to the k axis). Whenever $\sigma=\eta$ that locus consists of a vertical line at k^* (which is unique in that case). For $\sigma\neq\eta$ we can use (3.3) and the definition of μ and ξ to obtain the following expression for the $\dot{c}=0$ locus:

$$c(k)|_{\dot{c}=0}=f(k)\left(\frac{\eta}{\eta-\sigma}\right)\left(\frac{f'(k)\left(1-\frac{1}{\eta}\right)-(\delta+\rho)}{f'(k)-(\delta+\rho)}\right)$$

Given (3.3) consumption is increasing on any trajectory to the left of the $\dot{c}=0$, decreasing along trajectories to its right.[18]

Stationary equilibria correspond to intersections of the $\dot{c}=0$ and $\dot{k}=0$ loci on the (k,c) plane. Thus, in the case of a unique stationary equilibrium those loci interesect only once, regardless of the sign of $\sigma-\eta$. As a result, the equilibrium dynamics in that case are qualitatively identical to those of the neoclassical growth model: given k_0 there is a unique trajectory satisfying all the equilibrium conditions, that trajectory belongs to the stable manifold of the (unique) stationary equilibrium (which is a saddle[19]), and approaches the latter monotonically.

On the other hand, if multiple stationary equilibria exist (represented by multiple intersections of the $\dot{c}=0$ and $\dot{k}=0$ loci, as in Figures 2 and 3), the resulting dynamics will be substantially affected. That is the case we focus on in the remainder of the paper. Before we attempt a characterization of the global dynamics, we start by examining the behavior of equilibrium trajectories about each stationary equilibrium.

[18] When $c=f(k)$, it follows from (3 4) that consumption will be decreasing (along with the capital stock).
[19] See discussion in the next subsection.

5.1 Local stability properties

The stability properties of the nonlinear planar system (3.3)–(3.4) in a neighborhood of a given stationary equilibrium are established by examining the eigenvalues of the linearized dynamical system around it. That linearized system takes the form

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{pmatrix} f' - \delta & -1 \\ \left(\frac{1}{\gamma\mu}\right)\left(c^* f'' - \left(\frac{\mu'}{\mu}\right)\left(\frac{c^*}{k^*}\right)^2 \varepsilon^2\right) & \left(\frac{1}{\gamma\mu}\right)\left(\frac{\mu'}{\mu}\right)\left(\frac{c^*}{k^*}\right)\varepsilon \end{pmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix}$$

where μ, μ', f, f', f'', and $\varepsilon \equiv k^* \frac{f'}{f}$ are all evaluated at a given stationary equilibrium (k^*, c^*). Letting ζ_1 and ζ_2 denote the corresponding eigenvalues, it can be shown that

$$\zeta_1 \zeta_2 = R'(k^*)\left(\frac{c^*}{\gamma}\right) \tag{5.1}$$

$$\zeta_1 + \zeta_2 = f'(k^*) - \delta + \left(\frac{1}{\gamma\mu}\right)\left(\frac{\mu'}{\mu}\right)\left(\frac{c^*}{k^*}\right)\varepsilon \tag{5.2}$$

Thus, we see that the sign of $\zeta_1\zeta_2$ depends on the slope of $R(k^*)$ at the corresponding stationary equilibrium. If $R'(k^*) < 0$ (like k_l^* and k_h^* in Figure 1) the eigenvalues of the linearized system must be real and have opposite signs, implying that the associated stationary equilibrium is a saddle. Under our assumptions that will be a property of any stationary equilibrium that is (globally) unique, as well as of every "odd" stationary equilibrium in the presence of multiplicity.

On the other hand, "even" stationary equilibria (like k_m^* in Figure 1) satisfy $R'(k^*) > 0$, in which case (5.1) implies that the real part of the linearized system's eigenvalues must have the same sign. The stationary equilibrium will be a source (i.e., locally unstable) whenever $\zeta_1 + \zeta_2 > 0$, or a sink (i.e., locally stable) whenever $\zeta_1 + \zeta_2 < 0$.

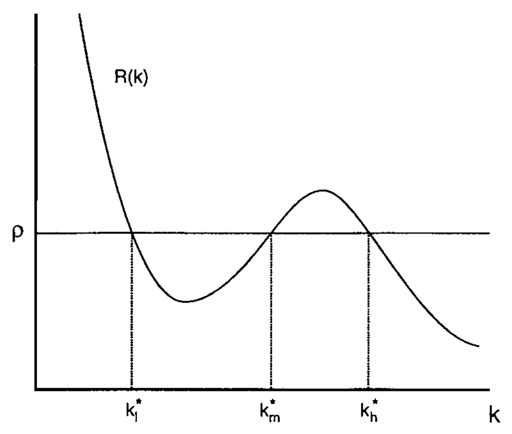

Figure 1

In order to illustrate the range of possible local dynamics around an "even" stationary equilibrium it is useful to examine how the eigenvalues of the linearized system change in response to variations in parameter γ. Such exercise is simplified by the fact that (k^*, c^*) is invariant to γ.[20] Define $\Omega(\gamma) \equiv \zeta_1 + \zeta_2$, and the discriminant of the characteristic equation $\Delta(\gamma) \equiv (\zeta_1 + \zeta_2)^2 - 4\zeta_1\zeta_2 = \Omega(\gamma)^2 - \left(\frac{1}{\gamma}\right) 4\, R'(k^*)c^*$, with all the terms evaluated at an "even" stationary equilibrium. Let $\gamma^* > 0$ be the (unique) solution to $\Omega(\gamma^*) = 0$. Clearly, $\Delta(\gamma^*) < 0$ in that case. Furthermore, it is easy to check the existence of a pair (γ_1, γ_2) satisfying $0 < \gamma_1 < \gamma^* < \gamma_2 < +\infty$, and such that $\Delta(\gamma_1) = \Delta(\gamma_2) = 0$, $\Delta(\gamma) > 0$ for all $\gamma \in (0, \gamma_1) \cup (\gamma_2, +\infty)$, and $\Delta(\gamma) < 0$ for all $\gamma \in [\gamma_1, \gamma_2]$. It follows that, as γ increases over its admissible range $(0, +\infty)$, the (even) stationary equilibrium becomes, consecutively, a stable node (all $\gamma \in (0, \gamma_1]$), a stable focus (all $\gamma \in (\gamma_1, \gamma^*)$), a center ($\gamma = \gamma^*$), an unstable focus (all $\gamma \in (\gamma^*, \gamma_2)$) and, finally, an unstable node (all $\gamma \in [\gamma_2, +\infty)$).

5.2 Global equilibrium dynamics

The stability properties discussed in the previous subsection are local, and thus pertain to the behavior of trajectories in a neighborhood of each stationary equilibrium. The interest of that analysis is somewhat limited for it leaves a number of important questions unanswered. Which of the stationary equilibria will an economy eventually approach (if any), given its initial capital stock? Does the initial capital stock determine the subsequent equilibrium trajectory (as in the neoclassical model)? Could multiple equilibrium paths coexist for a given initial capital stock? Answering those questions (and others) requires a characterization of the global dynamics. Given the local stability properties of stationary equilibria, such a characterization requires, in turn, knowledge of the "shape" on the (k, c) plane of the stable manifolds associated with odd – and, thus, saddle stable-stationary equilibria. Given the nonlinear nature of the system (3.3)–(3.4) a complete characterization of those manifolds in the presence of multiple stationary equilibria is a daunting task, one beyond the scope of this paper, even if we restrict ourselves to the case of three stationary equilibria. What I do instead is to present and discuss two types of qualitative global equilibrium dynamics that emerge in the presence of *three* stationary equilibria and which are illustrative of the qualitative implications of such multiplicity for off- steady state equilibrium paths.[21]

Figure 2 displays the equilibrium dynamics for an economy with an unstable middle stationary equilibrium. As discussed above that property is associated with a relatively high γ (low intertemporal substitution). Two of the trajectories departing from (k_m^*, c_m^*) coincide with two branches of the stable manifolds of (k_l^*, c_l^*) and (k_h^*, c_h^*), denoted respectively by W_l^s and W_h^s. The solid lines with arrows depict

[20] That invariance does no longer hold when exogenous technical progress is introduced in the model. Yet, the results presented here carry over to that case as long as the underlying growth rate is small. See Gali (1994b) for a discussion.

[21] An alternative route is taken in Gali (1994b), which discusses in detail the dynamics of a specific calibration of the model, obtained using numerical methods.

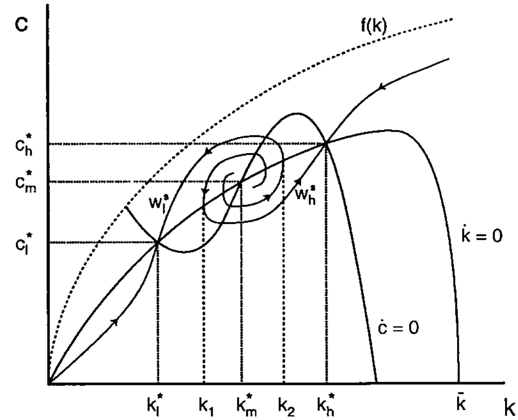

Figure 2

trajectories that satisfy all the equilibrium conditions. As the figure makes clear, the equilibrium set depends on the size of the initial capital stock relative to the benchmarks k_1 and k_2. The latter are defined by the projections onto the k axis of the leftmost point of W_h^s and the rightmost point of W_l^s, respectively. If $k(0) \in (0, k_1)$ the equilibrium is unique, and converges monotonically to the low stationary equilibrium (k_l^*, c_l^*). If $k(0) \in (k_2, \bar{k})$ the equilibrium is also unique but it converges now to (k_h^*, c_h^*) instead. Most interestingly, if $k(0) \in [k_1, k_2]$ there exist multiple trajectories that satisfy all the equilibrium conditions. Some of those paths lead to (k_l^*, c_l^*) whereas others converge to (k_h^*, c_h^*).[22] Thus, whether the economy converges to the high or the low stationary equilibrium depends on the selected path. That selection must be consistent with agents' (self-fulfilling) expectations as of $t = 0$ on the future path of the economy. In contrast with the neoclassical model, knowledge of the initial capital stock and the set of equilibrium conditions is no longer sufficient to pin down the equilibrium outcome.

The range of initial capital stock levels for which multiple equilibria exist depends on the extent of the overlap of the stable manifolds, which in turn depends in a complicated way on parameter values. In particular, the numerical simulations in Galí (1994b) suggest that the size of the multiplicity region $[k_1, k_2]$ tends to shrink as the value of γ increases (i.e., as we reduce the intertemporal substitution), and it eventually vanishes. When that occurs the equilibrium dynamics become determinate and have a threshold nature: if $k(0) < k_m^*$ the (unique) equilibrium path converges to (k_l^*, c_l^*); if $k(0) > k_m^*$ the (also unique) equilibrium path converges to (k_h^*, c_h^*). In other words, the stationary equilibrium to which an economy converges (and the trajectory followed) is completely determined by the initial capital stock.

Figure 3 displays the phase portrait of an economy with a stable middle stationary equilibrium, a feature associated with a relatively low γ value (i.e., high

[22] In the absence of limit cycles (see below), any trajectory which does not converge to any of the (interior) stationary equilibria and can be shown to violate either the transversality condition or the consumer's Euler equation. Accordingly, they can be ruled out as equilibria.

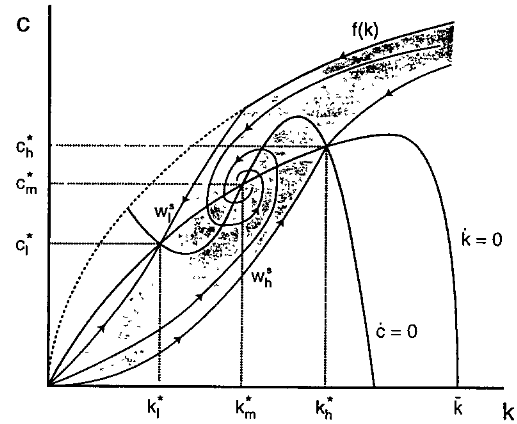

Figure 3

intertemporal substitution). For any given initial capital stock $k(0) \in (0, \bar{k})$ there are three types of trajectories consistent with a perfect foresight equilibrium. In the first type, the consumption level chosen as of $t = 0$ is the highest consistent with equilibrium (given $k(0)$) and the one that puts the economy exactly on W_l^s, implying subsequent convergence to the low stationary equilibrium. In the second type, agents choose the lowest possible consumption level compatible with equilibrium (given $k(0)$), i.e., the one on W_h^s; in that case the economy converges to the high stationary equilibrium. In addition to the previous trajectories, there exists a continuum of trajectories lying between W_l^s and W_h^s, represented by the shaded area. Each of those trajectories satisfies all the equilibrium conditions, eventually converging to the (stable) stationary equilibrium (k_m^*, c_m^*).

The kind of global equilibrium dynamics displayed in Figures 2 and 3 do not exhaust all the possibilities for the case with three stationary equilibria. In particular, a Hopf bifurcation occurs as γ crosses the imaginary axis at $\gamma = \gamma^*$, leading to one or more limit cycles about (k_m^*, c_m^*).[23] Furthermore, the configuration of W_l^s and W_h^s may be different in some cases from the ones depicted in Figures 2 and 3, though they will generally involve multiple equilibrium paths for some range of initial capital stock levels.

6 Discussion

What is the economic mechanism underlying the potential multiplicity of equilibrium paths in the model above? Recall that a demand elasticity that is increasing in the aggregate savings rate is a necessary condition for that multiplicity. Since, ceteris paribus, the private return to investment is increasing in the demand elasticity, the previous relationship leads to a complementarity between individual and aggregate decisions that lies at the root of the multiple equilibria, as in many other macro-

[23] See Galí (1994b) for a detailed discussion in the context of a numerical example. In the simulations reported there limit cycles emerge only for an extremely small range of γ values.

economic settings. The dynamic, perfect foresight nature of the model above introduces a further constraint on the set of trajectories that qualify as equilibria: given $k(0)$ the choice of an initial savings rate $\lambda(0)$ must lead to a future path of income and interest rates which induces, when perfectly foreseen, initial savings by individual consumers (and corresponding investment by firms) that precisely generate the aggregate savings rate $\lambda(0)$.

What kind of policy intervention could prevent the "coordination failure" associated with the low output equilibrium? Consider subsidizing firms' investment at a rate $\tau(t) = \mu(\lambda(t)) - 1 > 0$ (i.e., the subsidy rate is contingent on the aggregate savings rate $\lambda(t)$). With that subsidy scheme in place, the after-tax return on investment along a symmetric equilibrium, given by $\frac{1+\tau(t)}{\mu(\lambda(t))} f'(k)$, is equated to the marginal product of capital $f'(k)$. As a result, the μ term drops from equilibrium condition (3.3), and the efficient (Cass-Koopmans) allocation is restored as an equilibrium.

The possibility of multiple stationary equilibria in the growth model analyzed above has some interesting empirical implications.[24] First, the possible existence of different equilibrium trajectories leading to different steady states can in principle account for the evidence of no convergence in per capita income (e.g., Barro (1991) without having to rely on heterogenous preferences and technologies. Furthermore, and in contrast with the recent endogenous growth models, the possibility of no convergence between two economies obtains even if their initial conditions are identical, thus offering a potential explanation for so called "economic miracles" (Lucas (1993)). The model also implies the existence of "convergence clubs", i.e. economies with (possibly substantial) differences in initial conditions which eventually converge to the same stationary equilibrium (and thus the same income levels), in a way consistent with observations by Baumol et al. (1989). Finally, the model predicts that returns are asymptotically equalized across economies even in the absence of capital mobility. Such equalization of returns may coexist with large (and permanent) differences in capital and income levels. Thus, the model allows us to reconcile the absence of large (real) interest rate differentials across countries (and, the seeming failure of capital to flow from rich to poor countries) with the existence of potentially large (and persistent) income gaps.

Appendix

Solution to the consumer problem

The Hamiltonian associated with the second stage of the consumer problem is:

$$H(t) = (1-\gamma)^{-1}\left(\frac{e(t)}{P(t)}\right)^{1-\gamma} + \beta(t)\left(w(t) - e(t) + \int_0^1 (d_j(t) + \dot{q}_j(t))s_j(t)dj\right)$$
$$+ \phi(t)\beta(t)\left(a(t) - \int_0^1 q_j(t)s_j(t)dj\right)$$

[24] See Benhabib and Galí (1995) for a general discussion of the empirics of growth models with multiple equilibria.

Necessary conditions for a solution of that problem are given by:

$$\frac{\partial H(t)}{\partial e(t)} = 0 \rightarrow e(t)^{-\gamma} = \beta(t)P(t)^{1-\gamma} \tag{A1}$$

$$\frac{\partial H(t)}{\partial s_j(t)} = 0 \rightarrow \phi(t)q_j(t) = d_j(t) + \dot{q}_j(t) \tag{A2}$$

$$-\frac{\partial H(t)}{\partial a(t)} = \dot{\beta}(t) - \rho\beta(t) \rightarrow \dot{\beta}(t) = \beta(t)(\rho - \phi(t)) \tag{A3}$$

for all $j \in [0, 1]$. The previous conditions are sufficient if, in addition, the transversality condition $\lim_{T\to\infty} \beta(T)a(T)\exp(-\rho T)$ holds. 2.4 in the main text is obtained by integrating (A2) and (A3) forward and combining the result with (A1).

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