Additional material to the paper "Does money matter in shaping domestic business cycles? An international investigation"

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Appendix A: Derivation of the log-linearized conditions

Household's problem

The Lagrangian for the household's maximization problem is given by

$$\begin{aligned}\mathcal{L} &= E_0\Bigg\{\sum_{t=0}^{\infty}\beta^t\Bigg[a_t\left(U\left(x_t,\frac{M_t}{p_te_t}\right)-\eta n_t\right) \\ &+ \Lambda_t\left(M_{t-1}+T_t+B_{t-1}+W_tn_t+D_t-p_tc_t-\frac{B_t}{R_t}-M_t\right)\Bigg]\Bigg\}\end{aligned}$$

where $x_t \equiv c_t - hc_{t-1}$ is the flow of consumption good and h is an external habit formation parameter. Letting real balances be denoted by $m_t = M_t/p_t$, the first order conditions are

$$c_t: \quad a_tU_1(x_t, m_t/e_t) = \Lambda_t p_t \tag{1}$$

$$n_t: \quad a_t\eta = \Lambda_t W_t \tag{2}$$

$$B_t: \quad \frac{\Lambda_t}{R_t} = \beta E_t\Lambda_{t+1} \tag{3}$$

$$M_t: \quad \Lambda_t - a_tU_2(x_t, m_t/e_t)\frac{1}{p_te_t} = \beta E_t\Lambda_{t+1} \tag{4}$$

Letting $w_t = W_t/p_t$ denote the real wage rate and combining (1) and (2) yields the optimal labor-leisure condition

$$\eta = w_tU_1(x_t, m_t/e_t) \tag{5}$$

Letting $\pi_t = p_t/p_{t-1}$ denote the gross inflation rate in period t and combining (1) and (3) yields the intertemporal Euler equation

$$a_t U_1(x_t, m_t/e_t) = \beta R_t E_t \left[\frac{a_{t+1} U_1(x_{t+1}, m_{t+1}/e_{t+1})}{\pi_{t+1}} \right] \tag{6}$$

Finally, combining (1), (3) and (4) yields the money demand equation

$$(R_t - 1) e_t U_1(x_t, m_t/e_t) = R_t U_2(x_t, m_t/e_t) \tag{7}$$

Firm's problem

The first order condition from the firm's maximization of future discounted streams of real dividends is given by

$$\begin{aligned} 0 \;&=\; (1-\theta)\left(\frac{p_t^i}{p_t}\right)^{-\theta}\left(\frac{y_t}{p_t}\right) + \theta\left(\frac{p_t^i}{p_t}\right)^{-\theta-1}\left(\frac{y_t w_t}{z_t p_t}\right) - \phi\left(\frac{p_t^i}{\pi^s p_{t-1}^i} - 1\right)\left(\frac{y_t}{\pi^s p_{t-1}^i}\right) \\ &+\beta\phi E_t\left[\frac{a_{t+1}U_1(x_{t+1}, m_{t+1}/e_{t+1})}{a_t U_1(x_t, m_t/e_t)}\left(\frac{p_{t+1}^i}{\pi^s p_t^i} - 1\right)\left(\frac{y_{t+1}p_{t+1}^i}{\pi^s (p_t^i)^2}\right)\right] \end{aligned}$$

Since we consider a symmetric equilibrium in which all firms behave identically, $p_t^i = p_t$, the above equation simplifies to

$$\begin{aligned} 0 \;&=\; (1-\theta)\left(\frac{y_t}{p_t}\right) + \theta\left(\frac{y_t w_t}{z_t p_t}\right) - \phi\left(\frac{\pi_t}{\pi^s} - 1\right)\left(\frac{y_t \pi_t}{\pi^s p_t}\right) \\ &+\beta\phi E_t\left[\frac{a_{t+1}U_1(x_{t+1}, m_{t+1}/e_{t+1})}{a_t U_1(x_t, m_t/e_t)}\left(\frac{\pi_{t+1}}{\pi^s} - 1\right)\left(\frac{y_{t+1}\pi_{t+1}}{\pi^s p_t}\right)\right] \end{aligned}$$

Dividing by y_t/p_t and substituting for w_t from (5) yields

$$\begin{aligned} \theta - 1 \;&=\; \frac{\theta\eta}{z_t U_1(x_t, m_t/e_t)} - \phi\left(\frac{\pi_t}{\pi^s} - 1\right)\frac{\pi_t}{\pi^s} \\ &+\beta\phi E_t\left[\frac{a_{t+1}U_1(x_{t+1}, m_{t+1}/e_{t+1})}{a_t U_1(x_t, m_t/e_t)}\left(\frac{\pi_{t+1}}{\pi^s} - 1\right)\left(\frac{\pi_{t+1}y_{t+1}}{\pi^s y_t}\right)\right] \end{aligned} \tag{8}$$

Log-linearization

In what follows, a hat-variable denotes percentage-deviation from the steady state, for example, $\hat{R}_t = \log(R_t/R^s) \simeq \frac{dR_t}{R^s}$. Since all output is consumed in equilibrium, $y_t = c_t$. Furthermore, equation 6 in the steady state collapses to $1 = \beta\frac{R^s}{\pi^s}$. Finally, for notational convenience, we omit the arguments of the derivatives of the utility function , i.e. we will use U_1 for $U_1(x^s, m^s/e^s)$ and so on.

Euler equation

Log-linearizing (6) around a deterministic steady state we obtain

$$\begin{aligned} U_1(\hat{R}_t + E_t\hat{a}_{t+1} - E_t\hat{\pi}_{t+1}) + U_{11}y^s(E_t\hat{y}_{t+1} - h\hat{y}_t) + U_{12}\frac{m^s}{e^s}(E_t\hat{m}_{t+1} - E_t\hat{e}_{t+1}) \\ = U_1\hat{a}_t + U_{11}y^s\left(\hat{y}_t - h\hat{y}_{t-1}\right) + U_{12}\frac{m^s}{e^s}\left(\hat{m}_t - \hat{e}_t\right) \end{aligned}$$

Collecting terms gives

$$\begin{aligned} U_{11}y^s(1+h)\hat{y}_t &= U_{11}y^s E_t\hat{y}_{t+1} + U_{11}y^s h\hat{y}_{t-1} \\ &+ U_1[\hat{R}_t - E_t\hat{\pi}_{t+1} - (\hat{a}_t - E_t\hat{a}_{t+1})] \\ &- U_{12}\frac{m^s}{e^s}[(\hat{m}_t - \hat{e}_t) - (E_t\hat{m}_{t+1} - E_t\hat{e}_{t+1})] \end{aligned}$$

Defining

$$\omega_1 \equiv -\frac{U_1(x^s, m^s/e^s)}{y^s U_{11}(x^s, m^s/e^s)} \quad \text{and} \quad \omega_2 \equiv -\frac{(m^s/e^s)U_{12}(x^s, m^s/e^s)}{y^s U_{11}(x^s, m^s/e^s)}$$

the linearized Euler equation is

$$\begin{aligned} \hat{y}_t &= \frac{1}{1+h}E_t\hat{y}_{t+1} + \frac{h}{1+h}\hat{y}_{t-1} \\ &- \frac{\omega_1}{1+h}\left[\hat{R}_t - E_t\hat{\pi}_{t+1} - (\hat{a}_t - E_t\hat{a}_{t+1})\right] \\ &+ \frac{\omega_2}{1+h}\left[(\hat{m}_t - \hat{e}_t) - (E_t\hat{m}_{t+1} - E_t\hat{e}_{t+1})\right] \end{aligned} \tag{9}$$

Money demand

Linearizing the equation (7) yields

$$R^s e^s U_1 \hat{R}_t + (R^s - 1)e^s U_1 \hat{e}_t + (R^s - 1)e^s \left[y^s U_{11}(\hat{y}_t - h\hat{y}_{t-1}) + \frac{m^s}{e^s} U_{12}(\hat{m}_t - \hat{e}_t) \right]$$
$$= R^s U_2 \hat{R}_t + R^s y^s U_{21}(\hat{y}_t - h\hat{y}_{t-1}) + R^s m^s / e^s U_{22}(\hat{m}_t - \hat{e}_t)$$

In the steady state $(R^s - 1)e^s U_1 = R^s U_2$. Hence, collecting terms we obtain

$$\begin{aligned} &[(R^s - 1)e^s U_{12} - R^s U_{22}] \frac{m^s}{e^s} \hat{m}_t \\ &\qquad = [R^s U_{21} - (R^s - 1)e^s U_{11}] y^s (\hat{y}_t - h\hat{y}_{t-1}) - \frac{R^s}{R^s - 1} U_2 \hat{R}_t \\ &\qquad + \left[((R^s - 1)e^s U_{12} - R^s U_{22}) \frac{m^s}{e^s} - (R^s - 1)e^s U_1 \right] \hat{e}_t \end{aligned}$$

Defining

$$\begin{aligned} \gamma_2 &= \frac{R^s}{(R^s - 1)(m^s/e^s)} \left(\frac{U_2(x^s, m^s/e^s)}{(R^s - 1)e^s U_{12}(x^s, m^s/e^s) - R^s U_{22}(x^s, m^s/e^s)} \right) \\ \gamma_1 &= \left(R^s - 1 + R^s \omega_2 \frac{y^s}{m^s} \right) \left(\frac{\gamma_2}{\omega_1} \right) \end{aligned}$$

we can write the linearized money demand equation as

$$\hat{m}_t = \gamma_1(\hat{y}_t - h\hat{y}_{t-1}) - \gamma_2 \hat{R}_t + (1 - (R^s - 1)\gamma_2)\hat{e}_t \tag{10}$$

Phillips curve

When linearizing the optimal pricing rule (8), the equation simplifies considerably because the price adjustment cost term is zero in the steady state and we are left with $\theta - 1 = \frac{\theta\eta}{z^s U_1(x^s, m^s/e^s)}$. Hence

$$\begin{aligned} 0 &= -(\theta - 1)\hat{z}_t - (\theta - 1)\frac{U_{11} y^s}{U_1}(\hat{y}_t - h\hat{y}_{t-1}) - (\theta - 1)\frac{U_{12}\frac{m^s}{e^s}}{U_1}(\hat{m}_t - \hat{e}_t) \\ &\quad -\phi\hat{\pi}_t + \beta\phi E_t \hat{\pi}_{t+1} \end{aligned}$$

Solving for $\hat{\pi}_t$, defining $\psi = (\theta - 1)/\phi$ and using the definitions of ω_1 and ω_2:

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \psi \left(\frac{1}{\omega_1}(\hat{y}_t - h\hat{y}_{t-1}) - \frac{\omega_2}{\omega_1}(\hat{m}_t - \hat{e}_t) - \hat{z}_t \right) \tag{11}$$

Policy rule

Log-linearizing the assumed policy rule implies

$$\begin{aligned} \hat{R}_t &= \rho_r \hat{R}_{t-1} + (1-\rho_r)\rho_y \hat{y}_{t-p} + (1-\rho_r)\rho_\pi \hat{\pi}_{t-p} \\ &+ (1-\rho_r)\rho_m \Delta(\hat{m}_{t-p} + \hat{\pi}_{t-p}) + \hat{\epsilon}_t \end{aligned} \tag{12}$$

Summarizing, the log-linearized optimality conditions are

$$\begin{aligned} \hat{y}_t &= \frac{1}{1+h} E_t \hat{y}_{t+1} + \frac{h}{1+h} \hat{y}_{t-1} - \frac{\omega_1}{1+h} \left[\hat{R}_t - E_t \hat{\pi}_{t+1} - (\hat{a}_t - E_t \hat{a}_{t+1}) \right] \\ &+ \frac{\omega_2}{1+h} \left[(\hat{m}_t - \hat{e}_t) - (E_t \hat{m}_{t+1} - E_t \hat{e}_{t+1}) \right] \end{aligned} \tag{13}$$

$$\hat{m}_t = \gamma_1 (\hat{y}_t - h\hat{y}_{t-1}) - \gamma_2 \hat{R}_t + (1 - (R^s - 1)\gamma_2)\hat{e}_t \tag{14}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \psi \left(\frac{1}{\omega_1}(\hat{y}_t - h\hat{y}_{t-1}) - \frac{\omega_2}{\omega_1}(\hat{m}_t - \hat{e}_t) - \hat{z}_t \right) \tag{15}$$

$$\begin{aligned} \hat{R}_t &= \rho_r \hat{R}_{t-1} + (1-\rho_r)\rho_y \hat{y}_{t-p} + (1-\rho_r)\rho_\pi \hat{\pi}_{t-p} \\ &+ (1-\rho_r)\rho_m \Delta(\hat{m}_{t-p} + \hat{\pi}_{t-p}) + \hat{\epsilon}_t \end{aligned} \tag{16}$$

where

$$\omega_1 = -\frac{U_1(x^s, \frac{m^s}{e^s})}{y^s U_{11}(x^s, \frac{m^s}{e^s})} \tag{17}$$

$$\omega_2 = -\frac{m^s}{e^s} \frac{U_{12}(x^s, \frac{m^s}{e^s})}{y^s U_{11}(x^s, \frac{m^s}{e^s})} \tag{18}$$

$$\gamma_2 = \frac{R^s}{(R^s - 1)(m^s/e^s)} \left(\frac{U_2(x^s, \frac{m^s}{e^s})}{(R^s - 1)e^s U_{12}(x^s, \frac{m^s}{e^s}) - R^s U_{22}(x^s, \frac{m^s}{e^s})} \right) \tag{19}$$

$$\gamma_1 = (R^s - 1 + R^s \omega_2 \frac{y^s}{m^s})(\frac{\gamma_2}{\omega_1}) \tag{20}$$

$$\psi = \frac{\theta - 1}{\phi} \tag{21}$$

Appendix B: Estimates in alternative specifications

	Basic	$\omega_2 = 0$	$\rho_m = 0$	$\omega_2 = h = 0$	$\rho_m = h = 0$
ψ	2.0427	1.0055	0.0136	1.3679	1.9232
	(0.4450)	(0.2765)	(0.0102)	(0.3586)	(0.4926)
	0.9900	0.9900	0.9900	0.9900	0.9900
ω_2	-0.9794	0.0000	-0.1441	0.0000	-0.1987
	(0.2861)		(0.1940)		(0.1816)
γ_2	0.3647	0.4039	0.2854	0.3428	0.3520
	(0.0959)	(0.1044)	(0.0769)	(0.1020)	(0.0609)
h	0.1447	0.7324	0.8330	0.0000	0.0000
	(0.1006)	(0.0975)	(0.1431)		
ρ_r	0.6281	0.6408	0.7645	0.6375	0.6101
	(0.0378)	(0.0404)	(0.0979)	(0.0386)	(0.0412)
ρ_y	0.4769	0.5066	0.6811	0.4144	0.4584
	(0.1181)	(0.1049)	(0.0995)	(0.1024)	(0.1144)
ρ_p	6.0743	6.2789	0.4611	6.4268	6.8726
	(0.5099)	(0.5616)	(1.3736)	(0.5534)	(0.5774)
ρ_m	1.7993	1.8759	0.0000	1.8526	0.0000
	(0.2543)	(0.2789)		(0.2837)	
ρ_a	0.9862	0.9563	0.9327	0.9854	0.9878
	(0.0301)	(0.0156)	(0.0112)	(0.0007)	(0.0099)
ρ_e	0.9807	0.9732	0.9714	0.9800	0.9812
	(0.0332)	(0.0118)	(0.0111)	(0.0096)	(0.0126)
ρ_z	0.9517	0.6153	0.7805	0.9444	0.9560
	(0.0279)	(0.1048)	(0.1110)	(0.0336)	(0.0267)
σ_r	0.0092	0.0091	0.0098	0.0093	0.0103
	(0.0835)	(0.0902)	(0.0964)	(0.0761)	(0.0876)
σ_a	0.4849	0.1677	0.1164	0.4688	0.6066
	(3.5114)	(0.4051)	(0.2162)	(0.0926)	(1.4836)
σ_e	0.0088	0.0085	0.0091	0.0085	0.0083
	(0.0714)	(0.0742)	(0.0784)	(0.0657)	(0.0600)
σ_z	0.0029	0.0042	0.0417	0.0041	0.0030
	(0.1013)	(0.0989)	(0.3479)	(0.0888)	(0.0960)
Log L	1589.29	1578.42	1532.75	1573.88	1569.21

Table 1: Comparison with Ireland, US, 1980:1-2008:2. In the estimation β is calibrated at 0.99 and ω_1 at 2.

	M1	M0	Credit	SP500	House Prices	Oil	Goverment
ψ	1.9317	1.8982	1.4166	0.0533	0.0513	0.0761	0.3916
	(0.5031)	(0.2721)	(0.3635)	(0.0100)	(0.0222)	(0.0768)	(0.0505)
	0.9900	0.9900	0.9900	0.9900	0.9900	0.9900	0.9900
ω_2	-0.9454	-1.0499	-1.1056	0.3817	0.1307	0.1986	-0.4426
	(0.2646)	(0.2976)	(0.4825)	(0.3546)	(0.0895)	(1.5340)	(0.1223)
γ_2	0.3979	0.3373	0.0215	0.3431	1.0765	1.0870	0.7937
	(0.1428)	(0.0823)	(0.0389)	(0.4429)	(0.3796)	(2.9938)	(0.2274)
h	0.1446	0.1704	0.8380	0.8935	0.9610	0.9790	0.6957
	(0.1096)	(0.1193)	(0.0955)	(2.2193)	(0.2081)	(3.1747)	(0.0781)
ρ_r	0.6524	0.6475	0.6002	0.7819	0.5332	0.0033	0.6504
	(0.0406)	(0.0387)	(0.0417)	(0.1714)	(0.0811)	(0.1841)	(0.0409)
ρ_y	0.5959	0.5468	0.2831	1.1451	0.7566	0.2181	0.2347
	(0.1304)	(0.1188)	(0.1122)	(0.9329)	(0.1223)	(2.8471)	(0.0697)
ρ_p	6.6062	7.0230	7.1866	3.9549	4.3659	1.0174	4.0976
	(0.6091)	(0.6586)	(0.5458)	(4.3341)	(0.6476)	(2.3430)	(0.3428)
ρ_m	0.8487	1.7138	1.1908	0.9229	2.0508	1.8687	-0.1065
	(0.2186)	(0.2946)	(0.1874)	(1.9861)	(0.4496)	(3.1508)	(0.0485)
ρ_a	0.9897	0.9637	0.9673	0.1877	-0.1515	2.4511	0.9709
	(0.0007)	(0.0066)	(0.0235)	(0.1968)	(0.1847)	(3.0195)	(0.0140)
ρ_e	0.9893	0.9510	0.9706	0.9082	0.9310	0.9144	0.9529
	(0.0067)	(0.0097)	(0.0072)	(0.0435)	(0.0045)	(0.1397)	(0.0162)
ρ_z	0.9481	0.9172	0.4928	0.9856	0.9393	0.9546	0.7599
	(0.0297)	(0.0200)	(0.1377)	(0.0274)	(0.0332)	(0.2492)	(0.0554)
σ_r	0.0096	0.0093	0.0092	0.7903	0.8037	0.4226	0.0098
	(0.0825)	(0.0831)	(0.0791)	(0.0823)	(0.0778)	(1.3992)	(0.0688)
σ_a	0.6772	0.2034	0.2453	0.0075	0.1475	0.0054	0.1922
	(0.1855)	(0.2040)	(0.8852)	(0.7180)	(0.1206)	(2.6459)	(0.5538)
σ_e	0.0149	0.0107	0.0150	0.0086	0.0135	0.0261	0.0310
	(0.0680)	(0.0644)	(0.0722)	(0.1867)	(0.1691)	(2.4934)	(0.0531)
σ_z	0.0029	0.0028	0.0042	0.0975	0.1216	0.0996	0.0057
	(0.0925)	(0.0978)	(0.1042)	(0.5791)	(0.1024)	(2.2782)	(0.1022)
Log L	1516.48	1456.78	1514.66	1704.94	1869.09	1438.36	2324.89

Table 2: Robustness checks, US, sample 1980:1-2008:2. In the estimation β is calibrated at 0.99 and ω_1 at 2.

	$\omega_1 = 1$			$\omega_1 = 2$		
	Full sample	Sample 1	Sample 2	Full sample	Sample 1	Sample 2
ψ	0.1494	0.0175	0.5829	0.7433	0.0009	2.0427
	(0.0485)	(0.0110)	(0.1574)	(0.1595)	(0.0013)	(0.4450)
ω_2	-0.1155	-0.2313	-0.6372	-0.5112	24.1444	-0.9794
	(0.3202)	(0.3495)	(0.1257)	(0.4827)	(8.1261)	(0.2861)
γ_2	0.0881	0.0970	0.3259	0.0746	- 0.0278	0.3647
	(0.0871)	(0.1779)	(0.0813)	(0.0683)	(0.0283)	(0.0959)
h	0.6972	0.9554	0.2078	0.7169	0.8846	0.1447
	(0.0777)	(0.1010)	(0.1163)	(0.0696)	(0.0262)	(0.1006)
ρ_r	0.7190	0.6711	0.6408	0.6822	0.8080	0.6281
	(0.0376)	(0.0824)	(0.0329)	(0.0352)	(0.0707)	(0.0378)
ρ_y	0.2460	0.2769	0.4669	0.1774	0.3571	0.4769
	(0.0909)	(0.0882)	(0.1135)	(0.0690)	(0.0938)	(0.1181)
ρ_p	4.0873	2.3676	6.0247	4.3702	2.1510	6.0743
	(0.3095)	(0.3252)	(0.4626)	(0.3835)	(0.4814)	(0.5099)
ρ_m	1.4961	0.2139	1.7117	1.5788	-0.0363	1.7993
	(0.2490)	(0.3201)	(0.2578)	(0.1957)	(0.2187)	(0.2543)
ρ_a	0.9532	0.9866	0.9611	0.9630	0.9204	0.9862
	(0.0171)	(0.0042)	(0.0122)	(0.0060)	(0.0291)	(0.0301)
ρ_e	0.9773	0.9841	0.9746	0.9774	0.9689	0.9807
	(0.0087)	(0.0081)	(0.0117)	(0.0085)	(0.0002)	(0.0332)
ρ_z	0.7451	0.8395	0.9232	0.7005	0.8700	0.9517
	(0.0472)	(0.0806)	(0.0196)	(0.0623)	(0.0803)	(0.0279)
σ_r	0.0092	0.0071	0.0091	0.0099	0.0067	0.0092
	(0.0660)	(0.0865)	(0.0756)	(0.0672)	(0.1172)	(0.0835)
σ_a	0.1367	0.3990	0.1852	0.1625	0.1316	0.4849
	(0.4022)	(0.6678)	80.3140)	(0.1825)	(0.1683)	(3.5114)
σ_e	0.0092	0.0104	0.0089	0.0093	0.0110	0.0088
	(0.0526)	(0.0745)	(0.0687)	(0.0442)	(0.0676)	(0.0714)
σ_z	0.0128	0.0317	0.0066	0.0054	0.4911	0.0029
	(0.1092)	(1.0556)	(0.1232)	(0.0811)	(1.5415)	(0.1013)
Log L	2722.96	1165.78	1603.98	2674.93	1180.82	1589.29
Sample size	198	198	83	83	115	115

Table 3: Parameter estimates for US. In the estimation β is calibrated at 0.99.