How to write a higher order stochastic difference equation in Blanchard-Kahn form? The example I gave in class was a log-linearized Euler equation with a backward looking term; something that might come out of e.g. a model with habit formation in consumption.

$$\hat{c}_t = \theta E_t \hat{c}_{t+1} - \gamma \hat{c}_{t-1}$$

First, get rid of the expectations operator by defining an expectational error $\hat{c}_{t+1} \equiv E_t \hat{c}_{t+1} + \eta_{t+1}$

$$\hat{c}_t = \theta \hat{c}_{t+1} - \gamma \hat{c}_{t-1} - \theta \eta_{t+1}$$

Now, the problem is that the equation is second order, i.e. consumption appears both at time $t+1$ and at time $t-1$. To solve this problem, define a dummy variable

$$d_t = \hat{c}_{t-1}$$

Now, we can write the second order stochastic difference equation as a system of two first order equations.

$$\begin{aligned} \hat{c}_t &= \theta \hat{c}_{t+1} - \gamma d_t - \theta \eta_{t+1} \\ d_{t+1} &= \hat{c}_t \end{aligned}$$

Rearranging the first equation and forwarding the second equation by one period, we get the system in Blanchard-Kahn form.

$$\begin{aligned} \hat{c}_{t+1} &= \frac{1}{\theta} \hat{c}_t + \frac{\gamma}{\theta} d_t + \eta_{t+1} \\ d_{t+1} &= \hat{c}_t \end{aligned}$$

Or, in vector notation

$$\begin{bmatrix} \hat{c}_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 1/\theta & \gamma/\theta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ d_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{t+1} \\ 0 \end{bmatrix}$$