Log-linearizing

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1 What log-linearization is and how to do it

- Log-linearization means nothing else than linearization (i.e. first-order Taylor approximation) in logs.
  - Just like for linearization, the first question is "log-linearize around what?". We typically (log-)linearize our models around the steady state or (for more complicated models with frictions) around the frictionless steady state.
  - A technicality: if a variable has a zero steady state (like e.g. the nominal interest rate in a new Keynesian model), then we cannot log-linearize around the steady state for that variable. In that case, we normally log-linearize around 1+ the variable, so that the original variable reappears in the log-linearized model.

$$\widehat{1+i} = \log(1+i) - \log(1+\bar{\imath}) \simeq i$$

- Compared to linearization in levels log-linearization has some advantages:
  - Many models in economics are approximately log-linear, so log-linearization might be a better approximation than linearization.
  - The variables in the log-linearized model are easy to interpret because they are relative deviations from steady state.

$$\hat{x} = \log x - \log \bar{x} \simeq \frac{x - \bar{x}}{\bar{x}}$$

  - Parameters in the log-linearized model are easy to interpret because they are elasticities.
- There are several ways to log-linearize, some more cumbersome than others, depending on the specific equation:
  1. Linearize and substract and divide by steady state (example 1 below)
  2. Linearize in logs: let $x = \log X$, so that $X = e^x$ and linearize around x

– A variant of this is to substitute

$$x = e^{\log x} = e^{\log \bar{x}} e^{\log x - \log \bar{x}} = \bar{x} e^{\hat{x}}$$

for all variables and linearize around $\hat{x}$ (see example 2 below)

– It is sometimes useful to know some Taylor approximations by heart, in particular (Uhlig, see Canova p.53):

$$\begin{aligned} e^{\hat{x}} &\simeq 1 + \hat{x} \\ \hat{x}\hat{y} &\simeq 0 \end{aligned}$$

– Learn a few tricks by heart (see below)

- For all methods, it is important to realize that we can switch derivatives and integrals and are therefore allowed to (log)linearize inside the expectations operator.

1.1 Example 1

- Log-linearizing the Euler equation in the RBC model by linearizing and then substracting and dividing by the steady state.

  – The Euler equation

$$u'(c_t) = \beta E_t \left[A_{t+1} F_1 \left(K_{t+1}, h_{t+1} \right) u'(c_{t+1}) \right]$$

  – Linearizing left-hand side:

$$u'(c_t) \simeq u'(\bar{c}) + u''(\bar{c})(c_t - \bar{c})$$

  – Linearizing right-hand side:

$$\begin{aligned} & \beta E_t \left[A_{t+1} F_1 \left(K_{t+1}, h_{t+1} \right) u'(c_{t+1}) \right] \\ \simeq\ & \beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c}) + \beta F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c}) \left(E_t A_{t+1} - \bar{A} \right) \\ & + \beta u'(\bar{c}) \bar{A} \left[F_{11} \left(\bar{K}, \bar{h} \right) \left(E_t K_{t+1} - \bar{K} \right) + F_{12} \left(\bar{K}, \bar{h} \right) \left(E_t h_{t+1} - \bar{h} \right) \right] \\ & + \beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u''(\bar{c}) \left(E_t c_{t+1} - \bar{c} \right) \end{aligned}$$

  – By the original equation, the steady state terms drop out on both sides

  – Divide by steady state (left and right steady state are the same)

$$\begin{aligned} \frac{u''(\bar{c})}{u'(\bar{c})}(c_t - \bar{c}) \simeq\ & \frac{\beta F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c})}{\beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c})} \left(E_t A_{t+1} - \bar{A} \right) \\ & + \frac{\beta u'(\bar{c}) \bar{A}}{\beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c})} \left[F_{11} \left(\bar{K}, \bar{h} \right) \left(E_t K_{t+1} - \bar{K} \right) + F_{12} \left(\bar{K}, \bar{h} \right) \left(E_t h_{t+1} - \bar{h} \right) \right] \\ & + \frac{\beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u''(\bar{c})}{\beta \bar{A} F_1 \left(\bar{K}, \bar{h} \right) u'(\bar{c})} \left(E_t c_{t+1} - \bar{c} \right) \end{aligned}$$

– Making it look nice:

$$\frac{\bar{c}u''(\bar{c})}{u'(\bar{c})}\hat{c}_t \simeq E_t\hat{A}_{t+1}+\frac{\bar{K}F_{11}\left(\bar{K},\bar{h}\right)}{F_1\left(\bar{K},\bar{h}\right)}E_t\hat{K}_{t+1}+\frac{\bar{h}F_{12}\left(\bar{K},\bar{h}\right)}{F_1\left(\bar{K},\bar{h}\right)}E_t\hat{h}_{t+1}+\frac{\bar{c}u''(\bar{c})}{u'(\bar{c})}E_t\hat{c}_{t+1}$$

– With CRRA utility and Cobb Douglas production function

$$u(c)=\frac{1}{1-\theta}c^{1-\theta},\qquad F(K,h)=K^{\alpha}h^{1-\alpha}$$

$$-\theta\hat{c}_t \simeq E_t\hat{A}_{t+1}-(1-\alpha)\left(E_t\hat{K}_{t+1}-E_t\hat{h}_{t+1}\right)-\theta E_t\hat{c}_{t+1}$$

– Uff!

1.2 Example 2

- Same example, but linearizing in logs

  – Directly log-linearize (using Taylor approximation):

$$\begin{aligned}
u'\left(\bar{c}e^{\hat{c}_t}\right) &= \beta E_t\left[\bar{A}e^{\hat{A}_{t+1}}F_1\left(\bar{K}e^{\hat{K}_{t+1}},\bar{h}e^{\hat{h}_{t+1}}\right)u'\left(\bar{c}e^{\hat{c}_{t+1}}\right)\right]\\
\bar{c}^{-\theta}e^{-\theta\hat{c}_t} &= \beta E_t\left[\bar{A}e^{\hat{A}_{t+1}}\alpha\bar{K}^{-(1-\alpha)}e^{-(1-\alpha)\hat{K}_{t+1}}\bar{h}^{1-\alpha}e^{(1-\alpha)\hat{h}_{t+1}}\bar{c}^{-\theta}e^{-\theta\hat{c}_{t+1}}\right]\\
\bar{c}^{-\theta}e^{-\theta\hat{c}_t} &= \beta\bar{A}\alpha\bar{K}^{-(1-\alpha)}\bar{h}^{1-\alpha}\bar{c}^{-\theta}E_t\left[e^{\hat{A}_{t+1}}e^{-(1-\alpha)\hat{K}_{t+1}}e^{(1-\alpha)\hat{h}_{t+1}}e^{-\theta\hat{c}_{t+1}}\right]\\
e^{-\theta\hat{c}_t} &= E_t\left[e^{\hat{A}_{t+1}}e^{-(1-\alpha)\hat{K}_{t+1}}e^{(1-\alpha)\hat{h}_{t+1}}e^{-\theta\hat{c}_{t+1}}\right]\\
1-\theta\hat{c}_t &\simeq E_t\left[\left(1+\hat{A}_{t+1}\right)\left(1-(1-\alpha)\hat{K}_{t+1}\right)\left(1+(1-\alpha)\hat{h}_{t+1}\right)(1-\theta\hat{c}_{t+1})\right]\\
&\simeq E_t\left[1+\hat{A}_{t+1}-(1-\alpha)\hat{K}_{t+1}+(1-\alpha)\hat{h}_{t+1}-\theta\hat{c}_{t+1}\right]\\
-\theta\hat{c}_t &\simeq E_t\hat{A}_{t+1}-(1-\alpha)\left(E_t\hat{K}_{t+1}-E_t\hat{h}_{t+1}\right)-\theta E_t\hat{c}_{t+1}
\end{aligned}$$

  – Note: it looks like we could have just taken logs, but then how do we deal with the expectation operator? So, even though the original equation is multiplicative, this is still an approximation!

2 Some useful tricks

1. $z=xy\Rightarrow\hat{z}=\hat{x}+\hat{y}$
2. $z=x/y\Rightarrow\hat{z}=\hat{x}-\hat{y}$
3. $z=ax\Rightarrow\hat{z}=\hat{x}$
4. $z=x^a\Rightarrow\hat{z}=a\hat{x}$
5. $z=x+y\Rightarrow\hat{z}=\frac{\bar{x}}{\bar{z}}\hat{x}+\frac{\bar{y}}{\bar{z}}\hat{y}$

- These are just tricks that you may want to learn by heart in order to save time. There is nothing mystical about them: all are easily proven by linearizing and dividing by the steady state.

2.1 Example 3

- This is the neoclassical growth model from question 1 in the first problem set. I log-linearize the equations using the tricks.
  - Euler equation

$$\begin{aligned}\hat{c}_{t+1} &= \widehat{\beta\alpha A k_{t+1}^{-(1-\alpha)} c_t} \\ &= \widehat{k_{t+1}^{-(1-\alpha)} c_t} \text{ by trick 3} \\ &= \widehat{k_{t+1}^{-(1-\alpha)}} + \hat{c}_t \text{ by trick 1} \\ &= -(1-\alpha)\hat{k}_{t+1} + \hat{c}_t \text{ by trick 4}\end{aligned}$$

  - Resource constraint

$$\begin{aligned}\hat{k}_{t+1} &= \widehat{Ak_t^\alpha - c_t} \\ &= \frac{\bar{y}}{\bar{y}-\bar{c}}\widehat{Ak_t^\alpha} + \frac{-\bar{c}}{\bar{y}-\bar{c}}\widehat{-c_t} \text{ by trick 5}\end{aligned}$$

    where $\bar{y} = A\bar{k}^\alpha$

$$\begin{aligned}\hat{k}_{t+1} &= \frac{\bar{y}}{\bar{y}-\bar{c}}\widehat{k_t^\alpha} - \frac{\bar{c}}{\bar{y}-\bar{c}}\widehat{c_t} \text{ by trick 3} \\ &= \frac{\alpha\bar{y}}{\bar{y}-\bar{c}}\widehat{k_t} - \frac{\bar{c}}{\bar{y}-\bar{c}}\widehat{c_t} \text{ by trick 4}\end{aligned}$$

    Often, we like to simplify notation a bit by writing something like

$$\hat{k}_{t+1} = \alpha(1+\gamma)\widehat{k_t} - \gamma\widehat{c_t}$$

    where $\gamma = \bar{c}/(\bar{y}-\bar{c})$.