Partial equilibrium search in discrete time

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Here, I derive the solution for the partial equilibrium search model, as in RSW section 3.1, in discrete time.

1 Derivation in discrete time

The value function of the worker is given by $W(w)$ if she is employed in a job with wage w, and U if she is unemployed. Once employed, the worker does not have any choice anymore, and the Bellman equations for an employed worker is given by

$$W(w) = w + \beta(1-\lambda)W(w) + \beta\lambda U$$

where w is the flow payoff (the wage) and $(1-\lambda)W + \lambda U$ is the expected continuation value, which consists of remaining employed with probability $1-\lambda$ and becoming unemployed with probability λ. The probability λ that the job breaks, is also called the separation rate. An unemployed worker receives flow payoff b (the unemployment benefit) and has a choice whether to accept or reject tomorrow's wage offer.

$$U = b + \beta E_t\left[\max\{U, W(w_{t+1})\}\right] = b + \beta\int_0^\infty \max\{U, W(\omega)\}\, dF(\omega)$$

This expression is the same as in the model with a zero separation rate. Notice that we should be careful to distinguish between the realized wage w and the integration dummy ω.

We guess that the optimal policy is to accept job offers if and only if the wage is above a reservation wage, $w \geq w_R$. If the value function for an employed worker $W(w)$ is strictly increasing in the wage offer w (to be verified) and w_R such that $W(w_R) = U$, then this policy guarantees that jobs are accepted if and only if $W(w) \geq U$, which is optimal. It remains to find an expression for the reservation wage.

First, rewrite the RHS of both Bellman equations in terms of the value of being unemployed U and the surplus value of being employed $W(w) - U$.

$$\begin{aligned}(1-\beta)W(w) &= w - \beta\lambda\left[W(w) - U\right] \\ (1-\beta)U &= b + \beta\int_0^\infty \max\{0, W(w) - U\}\, dF(w)\end{aligned}$$

Then, substract the BE for U from the BE for $W(w)$ to get an expression for $W(w) - U$,

$$(1-\beta)\left[W(w) - U\right] = w - b - \beta\lambda\left[W(w) - U\right] - \beta\int_0^\infty \max\{0, W(\omega) - U\}\, dF(\omega)$$

$$(1-\beta+\beta\lambda)\left[W(w)-U\right]=w-b-\beta\int_0^\infty \max\{0,W(\omega)-U\}\,dF(\omega)$$

Evaluating this expression for $w=w_R$ and using that $W(w_R)=U$ gives:

$$(1-\beta+\beta\lambda)\left[W(w_R)-U\right]=0=w_R-b-\beta\int_0^\infty \max\{0,W(\omega)-U\}\,dF(\omega)$$

$$w_R=b+\beta\int_0^\infty \max\{0,W(\omega)-U\}\,dF(\omega)$$

Substituting back into the BE for $W(w)-U$:

$$\begin{aligned}(1-\beta+\beta\lambda)\left[W(w)-U\right] &= w-b-\beta\int_0^\infty \max\{0,W(\omega)-U\}\,dF(\omega)\\ &= w-b-(w_R-b)=w-w_R\end{aligned}$$

$$W(w)-U=\frac{w-w_R}{1-\beta+\beta\lambda}$$

And substituting back into the expression for the reservation wage:

$$\begin{aligned}w_R &= b+\beta\int_0^\infty \max\{0,W(\omega)-U\}\,dF(\omega)\\ &= b+\frac{\beta}{1-\beta+\beta\lambda}\int_0^\infty \max\{0,w-w_R\}\,dF(\omega)\\ &= b+\frac{\beta}{1-\beta+\beta\lambda}\int_{w_R}^\infty (w-w_R)\,dF(\omega)\\ &= b+\frac{1}{r+\lambda}\int_{w_R}^\infty (w-w_R)\,dF(\omega)\end{aligned}$$

where the last step follows from $\beta=1/(1+r)$.

2 Link to continuous time

Had we assumed that "wages and unemployment benefits are paid at the end of the period", so that current flow payoffs are discounted, then the algebra would have simplified a bit and all intermediate steps would have been the same as in the paper as well. Let $w=\beta\tilde{w}$ and $b=\beta\tilde{b}$.

Bellman equation employed worker

$$\begin{aligned}W(\tilde{w}) &= \beta\tilde{w}+\beta(1-\lambda)W(\tilde{w})+\beta\lambda U\\ (1+r)W(\tilde{w}) &= \tilde{w}+(1-\lambda)W(\tilde{w})+\lambda U\\ (r+\lambda)W(\tilde{w}) &= \tilde{w}+\lambda U\end{aligned}$$

Bellman equation unemployed worker:

$$\begin{aligned}U &= \beta\tilde{b}+\beta\int_0^\infty \max\{U,W(\tilde{\omega})\}\,dF(\tilde{\omega})\\ (1+r)U &= \tilde{b}+\int_0^\infty \max\{U,W(\tilde{\omega})\}\,dF(\tilde{\omega})\\ rU &= \tilde{b}+\int_0^\infty \max\{0,W(\tilde{\omega})-U\}\,dF(\tilde{\omega})\end{aligned}$$

Rewrite BE employed worker:

$$rW\left(\tilde{w}\right)=\tilde{w}-\lambda\left[W\left(\tilde{w}\right)-U\right]$$

Substract BE unemployed worker:

$$r\left[W\left(\tilde{w}\right)-U\right]=\tilde{w}-\tilde{b}-\lambda\left[W\left(\tilde{w}\right)-U\right]-\int_{0}^{\infty}\max\left\{0,W\left(\tilde{\omega}\right)-U\right\}dF\left(\tilde{\omega}\right)$$

$$\left(r+\lambda\right)\left[W\left(\tilde{w}\right)-U\right]=\tilde{w}-\tilde{b}-\int_{0}^{\infty}\max\left\{0,W\left(\tilde{\omega}\right)-U\right\}dF\left(\tilde{\omega}\right)$$

Evaluating in $\tilde{w}=\tilde{w}_R$ and using $W\left(\tilde{w}_R\right)=U$:

$$\tilde{w}_R=\tilde{b}+\int_{0}^{\infty}\max\left\{0,W\left(\tilde{\omega}\right)-U\right\}dF\left(\tilde{\omega}\right)$$

Substituting back into BE for $W\left(\tilde{w}\right)-U$:

$$W\left(\tilde{w}\right)-U=\frac{\tilde{w}-\tilde{w}_R}{r+\lambda}$$

And back into the expression for the reservation wage:

$$\tilde{w}_R=\tilde{b}+\frac{1}{r+\lambda}\int_{\tilde{w}_R}^{\infty}\left(\tilde{\omega}-\tilde{w}_R\right)dF\left(\tilde{\omega}\right)$$

This expression is the same as above, and -after some manipulations- as in equation (18) in the paper (apart from the α parameter). Notice that $\beta\tilde{w}_R=w_R$, which explains why the expression is the same even though it has $\tilde{b}$ and $\tilde{w}$ rather than b and w.