Monetary Policy and Bubbles in a New Keynesian Model with Overlapping Generations

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Abstract

I develop an extension of the New Keynesian model that features overlapping generations of finitely-lived agents. In contrast with the standard model, the proposed framework allows for the existence of rational expectations equilibria with asset price bubbles. I examine the conditions under which bubbly equilibria may emerge and the implications for the design of monetary policy. Monetary policies that lean against the bubble are shown to be potentially destabilizing, and likely to be dominated by inflation targeting policies.

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The rise and eventual collapse of speculative bubbles are viewed by many economists and policymakers as an important source of macroeconomic instability. A monetary policy that focuses narrowly on inflation and output stability but which neglects the emergence and rapid growth of asset bubbles is often perceived as a potential risk to medium-term macroeconomic and financial stability.[1]

Interestingly, the recurrent reference to bubbles in the policy debate contrasts with their conspicuous absence in modern monetary models. A likely explanation for this seeming anomaly lies in the fact that standard versions of the New Keynesian model, the workhorse framework used in monetary policy analysis, leave no room for the existence of bubbles in equilibrium, and hence for any meaningful *model-based* discussion of their possible interaction with monetary policy.[2]

In the present paper I develop a modified version of the New Keynesian model featuring overlapping generations of finitely-lived consumers and retirement.[3] For brevity, I henceforth refer to that framework as the OLG-NK model. The assumption of an infinite sequence of generations makes it possible for the transversality condition of any *individual* consumer to be satisfied in equilibrium, even in the presence of a bubble that grows at the rate of interest.[4] On the other hand, the assumption of retirement (or, more generally, of an eventual transition to inactivity) can generate an equilibrium rate of interest below the economy's trend growth rate, which is a condition necessary for the size of the bubble to remain bounded relative to the size of the economy. Finally, the assumption of sticky prices –a key feature of the New Keynesian model– has two important implications that are missing in most models with bubbles found in the literature. Firstly, price stickiness makes it possible for an aggregate bubble to influence aggregate demand and, hence, output and employment. Secondly, price stickiness makes monetary policy non-neutral, allowing it to counteract bubble-drive fluctuations and, under some conditions, to influence the size of the bubble itself. An appealing feature of the OLG-NK framework is that it nests the standard New

[1] See, e.g., Borio and Low (2002) for an early statement of that view. Taylor (2014) points to excessively low interest rates in the 2000s as a factor behind the housing boom that preceded the financial crisis of 2007-2008.

[2] The reason is well known: the equilibrium requirement that the bubble grows at the rate of interest violates the transversality condition of the infinite-lived representative consumer assumed in the New Keynesian model (as well as most macro models). See, e.g., Santos and Woodford (1997).

[3] Other authors have extended recently the New Keynesian model to incorporate overlapping generations of finite-lived agents into the New Keynesian framework, though none of them has allowed for the existence of bubbles. See literature discussion below.

[4] And even though that transversality condition does not hold for the economy as a whole.

Keynesian model (the NK model, henceforth) as a limiting case, when the probability of death and that of retirement approach zero.

After deriving the equations describing the model's equilibrium, I characterize the balanced growth paths consistent with that equilibrium and discuss the conditions under which a non-vanishing bubble may exist along those paths. If the incidence of retirement is sufficiently low (relative to the consumer's discount rate), there exists a unique balanced growth path, and it is a bubbleless one (as in the standard model). On the other hand, if the probability of retirement is sufficiently high (but plausibly so), a multiplicity of bubbly balanced growth paths is shown to exist, in addition to a bubbleless one (which always exists).

Once I characterize the existence and potential multiplicity of balanced growth paths –bubbly and bubbleless– I turn to the analysis of the stability properties of those paths and the role of monetary policy in determining those properties. In order to do so, I log-linearize the equilibrium conditions around a balanced growth path, as in the standard analysis of the textbook NK model, underscoring throughout the tractability of the extended framework.

Several findings of interest emerge from that analysis. First, I show the possibility of bubble-driven fluctuations in a neighborhood of a balanced growth path. I show that, in contrast with the standard NK model, the conventional Taylor principle generally fails to guarantee the local uniqueness of the equilibrium. In particular, stationary fluctuations in output and inflation may arise in association with fluctuations in the size of the bubble, even in the absence of fundamental shocks. I illustrate the possibility of such fluctuations by simulating calibrated versions of the OLG-NK.

Alternative monetary policy strategies to prevent or counteract bubble-driven fluctuations in the output gap and inflation are proposed and discussed. Two of these strategies involve a direct interest rate response to fluctuations in the aggregate bubble and require either an unrealistically large response to the bubble or a highly accurate one. Mismeasurement of the bubble or an inaccurate "calibration" of the policy response to its fluctuations may end up destabilizing output and inflation, as well as the bubble itself. By way of contrast, I show that a policy that targets inflation directly may attain the same stabilization objectives without the risks associated with direct responses to the bubble.

The paper concludes with some reflections on some of the caveats and limitations of the NK model developed here as a framework to capture the role played by bubbles as a source of economic fluctuations.

The rest of the paper is organized as follows. The next section summarizes the related literature. Section 2 describes the basic framework underlying the analysis in the rest of the paper. Section 3 characterizes the economy's balanced growth paths. Section 4 analyzes the equilibrium dynamics in a neighborhood of a balanced growth path, and the role of monetary policy in preventing indeterminacy of equilibria. Section 5 discusses the consequences of alternative policies in the face of bubble-driven fluctuations. Section 6 summarizes and concludes.

1 Related Literature

Much of the literature on rational bubbles in general equilibrium has been based on real models. An early reference in that category is Tirole (1985), using a conventional overlapping generations (OLG) framework with capital accumulation. A more recent one is Martín and Ventura (2012), who modify the Tirole model by introducing financial frictions that are alleviated by the existence of a bubble.

There is also an extensive literature on bubbles using monetary models with fully flexible prices. In most of those models, including the seminal paper by Samuelson (1958), money itself is the bubbly asset. Asriyan et al. (2016) provide a more recent example, introducing the notion of a nominal bubble. While monetary policy is not always neutral in those models, the mechanism through which its effects are transmitted is very different from that emphasized in models with nominal rigidities.

A number of papers have modified the standard NK model by introducing overlapping generations à la Blanchard-Yaari, though none of them has considered the possibility of bubbles. Piergallini (2006) develops a related model with money in the utility function to analyze the implications of the real balance effect on the stability properties of interest rate rules. Nisticò (2012) discusses the desirability of a systematic monetary policy response to stock price developments in a similar model, but in the absence of bubbles. Del Negro, Giannoni and Patterson (2015) propose a related framework as a possible solution to the "forward guidance puzzle." None of the previous authors allow for retirement or declining labor income in their frameworks. That feature plays a central role in the emergence of asset price bubbles in the model proposed here.

Bernanke and Gertler (1999, 2001) analyze the possible gains from "leaning against the wind" monetary policies in a NK model in which stock prices contain an *ad-hoc* deviation from their fundamental value. The properties of that deviation differ from those of a rational bubble, which cannot exist in their model, which assumes an infinitely-lived representative consumer.

In Galí (2014) I studied the interaction between rational bubbles and monetary policy in a two-period overlapping generations model with sticky prices, emphasizing some of the risks associated with "leaning against the bubble" policies. While closest in spirit to the present paper, the framework used in that paper had important limitations. In particular, it ruled out the possibility of bubble-driven fluctuations, since employment and output were constant in equilibrium, with the bubble only having redistributive effects. On the other hand, the assumption of two-period lived individuals made in Galí (2014), while convenient, cannot be easily reconciled with the frequency of observed asset boom-bust episodes (not to say with the observed duration of individual prices). By way of contrast, the model developed here displays endogenous fluctuations in output and employment in response to fluctuations in asset price bubbles, and it is consistent with a calibration of the model to a quarterly frequency (as is convention in the business cycle literature). Finally, an additional advantage of the framework developed below is that it nests the standard NK model as a limiting case.[5]

2 A New Keynesian Model with Overlapping Generations

Next I describe the basic framework underlying the analysis in the rest of the paper.

[5] Two recent working papers have explored, using alternative perspectives, the connection between monetary policy and asset bubbles. In contrast with the present contribution, however, both papers involve frameworks characterized by a monetary transmission mechanism very different from that found in the standard NK model. Thus, Allen et al. (2017) revisit the relationship between interest rates and asset bubbles discussed in Gali (2014) using a variety of frameworks (mostly non-monetary), and exploring the conditions and environments that determine the sign of that relation. Dong et al. (2017) analyze the implications of alternative monetary policy rules (including rules that respond systematically to asset bubbles) in an economy with infinitely-lived agents, and where a bubbly asset can help alleviate entrepreneurs' funding constraints. Monetary policy affects the amount of liquidity in the economy –and, as a result, on the value attached to the bubbly asset– through the impact of inflation real reserves.

2.1 Consumers

I assume an economy with overlapping generations of the "perpetual youth" type, as in Yaari (1965) and Blanchard (1984). The size of the population is constant and normalized to one. Each individual has a constant probability γ of surviving into the following period, independently of his age and economic status ("active" or "retired"). A cohort of size $1-\gamma$ is born (in an economic sense) and becomes active each period. Thus, the size in period $t \geq s$ of a cohort born in period s is given by $(1-\gamma)\gamma^{t-s}$.

At any point in time, active and retired individuals coexist in the economy. Active individuals supply labor and manage their own firms, which they set up when they join the economy. I assume that each active individual faces a constant probability $1-\upsilon$ of becoming "inactive." i.e. of permanently losing his job and quitting his entrepreneurial activities. That probability is independent of his age. For convenience, below I refer to that transition as "retirement," though it should be clear that it can be given a broader interpretation.[6]

The previous assumptions imply that the size of the active population (and, hence, the measure of firms) at any point in time is constant and given by $\alpha \equiv (1-\gamma)/(1-\upsilon\gamma) \in (0,1]$.

A representative consumer from cohort s, standing in period 0, chooses a consumption plan to maximize expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} (\beta\gamma)^t \log C_{t|s}$$

subject to the sequence of period budget constraints

$$\frac{1}{P_t} \int_0^{\alpha} P_t(i) C_{t|s}(i) di + E_t\{\Lambda_{t,t+1} Z_{t+1|s}\} = A_{t|s} + W_t N_{t|s} \tag{1}$$

for $t = 0, 1, 2, ...$ $\beta \equiv 1/(1+\rho) \in (0,1)$ is the discount factor. $C_{t|s} \equiv \left(\alpha^{-\frac{1}{\epsilon}} \int_0^{\alpha} C_{t|s}(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$ is a consumption index, with $C_{t|s}(i)$ being the quantity purchased of good $i \in [0,\alpha]$, at a price $P_t(i)$. $P_t \equiv \left(\alpha^{-1} \int_0^{\alpha} P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$ is the price index.

Complete markets for state-contingent securities are assumed, with $Z_{t+1|s}$ denoting the stochastic payoff (expressed in units of the consumption index) generated by a portfolio of securities

[6] Gertler (1999) introduces retirement in a similar fashion in a model of social security. More recently, Carvalho et al (2016) have used a version of the Gertler model to analyze the sources of low frequency changes in the equilibrium real rate. Both papers develop *real* models, in contrast to the present one, and do not consider the possibility of bubbles.

purchased in period t, with value given by $E_t\{\Lambda_{t,t+1} Z_{t+1|s}\}$, where $\Lambda_{t,t+1}$ is the stochastic discount factor for one-period-ahead (real) payoffs. Only individuals who are alive can trade in securities markets. Note that the existence of complete securities markets allows individuals to insure against the "risk" of retirement.

Variable $A_{t|s}$ denotes financial wealth at the start of period t, for a member of cohort $s \leq t$. For individuals other than those joining the economy in the current period, $A_{t|s} = Z_{t|s}/\gamma$, where the term $1/\gamma$ captures the additional return on wealth resulting from an annuity contract. As in Blanchard (1984), that contract has the holder receive each period from a (perfectly competitive) insurance firm an annuity payment proportional to his financial wealth, in exchange for transferring the latter to the insurance firm upon death.[7]

Variable W_t denotes the (real) wage per hour, and $N_{t|s}$ is the number of individual work hours. Both the wage and work hours are taken as given by each worker. Work hours employed are determined by firms and allocated uniformly among all active individuals, i.e. $N_{t|s} = N_t/\alpha$, where N_t is aggregate employment.[8] Note that $N_{t|s} = 0$ for retired individuals. Normalizing an active individual's time endowment to unity, it must be the case that $N_{t|s} \leq 1$ for all t and s, which I assume throughout.

Finally, I assume a solvency constraint of the form $\lim_{T\to\infty} \gamma^T E_t\{\Lambda_{t,t+T} A_{t+T|s}\} \geq 0$ for all t, where $\Lambda_{t,t+T}$ is determined recursively by $\Lambda_{t,t+T} = \Lambda_{t,t+T-1}\Lambda_{t+T-1,t+T}$.[9]

The problem above yields a set of optimal demand functions

$$C_{t|s}(i) = \frac{1}{\alpha}\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} C_{t|s} \tag{2}$$

for all $i \in [0, \alpha]$, which in turn imply $\int_0^\alpha P_{t|s}(i) C_{t|s}(i) di = P_t C_{t|s}$. Thus we can rewrite the period budget constraint as:

$$C_{t|s} + \gamma E_t\{\Lambda_{t,t+1} A_{t+1|s}\} = A_{t|s} + W_t N_{t|s} \tag{3}$$

[7] Thus, individuals who hold negative assets will pay an annuity fee to the insurance company. The latter absorbs the debt in case of death. The insurance arrangement can also be replicated through securities markets. In that case the individual will purchase a portfolio that generates a random payoff $A_{t+1|s}$ if he remains alive, 0 otherwise. The value of that payoff will be given by $E_t\{\Lambda_{t,t+1}\gamma A_{t+1|s}\}$ which is equivalent to the formulation in the main text, given that $A_{t|s} = Z_{t|s}/\gamma$.

[8] By not including hours of work in the utility function I effectively eliminate any wealth effects that would generate systematic counterfactual differences in the quantity of labor supplied by active individuals across age groups. Alternatively one may assume preferences that rule out those wealth effects, but at the cost of rendering the analysis below less tractable.

[9] Note that $(\Lambda\gamma)^{-1}$ is the "effective" interest rate paid by a borrower in the steady state. The solvency constraint thus has the usual interpretation of a no-Ponzi game condition.

The consumer's optimal plan must satisfy the optimality condition[10]

$$\Lambda_{t,t+1} = \beta \frac{C_{t|s}}{C_{t+1|s}} \tag{4}$$

and the transversality condition

$$\lim_{T \to \infty} \gamma^T E_t \left\{ \Lambda_{t,t+T} A_{t+T|s} \right\} = 0 \tag{5}$$

with (4) holding for all possible states of nature (conditional on the individual remaining alive in $t+1$).

The absence of labor disutility implies that each consumer will be willing to supply up to one unit of labor (his time endowment) every period. But as discussed above the quantity of labor effectively hired is determined by firms

2.2 Firms

Each individual is endowed with the know-how to produce a differentiated good, and sets up a firm with that purpose when he joins the economy. That firm remains operative until its founder retires or dies, whatever comes first.[11] All firms have an identical technology, represented by the linear production function

$$Y_t(i) = \Gamma^t N_t(i) \tag{6}$$

where $Y_t(i)$ and $N_t(i)$ denote output and employment for firm $i \in [0, \alpha]$, respectively, and $\Gamma \equiv 1 + g \geq 1$ denotes the (gross) rate of productivity growth. Individuals cannot work at their own firms, and must hire instead labor services provided by others.[12]

Aggregation of (2) across consumers yields the demand schedule facing each firm

$$C_t(i) = \frac{1}{\alpha} \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t$$

[10] Note that in the optimality condition the probability of remaining alive γ and the extra return $1/\gamma$ resulting from the annuity contract cancel each other. Complete markets guarantee the same consumption growth rate between two different periods for all consumers alive in the two periods, including those that are becoming retired.

[11] The assumption of finite-lived firms (or more generally, for firms whose dividends shrink relative to the size of the economy) is needed in order for bubbles to exist in equilibrium. By equating the probability of a firm's survival to that of its owner remaining alive and active I effectively equate the rate at which dividends and labor income are discounted, which simplifies considerably the analysis below.

[12] I assume that each firm newly set up in any given period inherits (through random assignment) the index of an exiting firm.

where $C_t \equiv (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s}$ is aggregate consumption. Each firm takes as given the aggregate price level P_t and aggregate consumption C_t.

As in Calvo (1983), each firm is assumed to freely set the price of its good with probability $1-\theta$ in any given period, independently of the time elapsed since the last price adjustment. With probability θ, an incumbent firm keeps its price unchanged, while a newly created firm sets a price equal to the economy's average price in the previous period.[13] Accordingly, the aggregate price dynamics are described by the equation

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}$$

where P_t^* is the price set in period t by firms optimizing their price.[14] A log-linear approximation of the previous difference equation around the zero inflation equilibrium yields (letting lower case letters denote the logs of the original variables):

$$p_t = \theta p_{t-1} + (1-\theta)p_t^* \tag{7}$$

i.e. the current price level is a weighted average of last period's price level and the newly set price, all in logs, with the weights given by the fraction of firms that do not and do adjust prices, respectively.

In both environments, a firm adjusting its price in period t will choose the price P_t^* that maximizes

$$\max_{P_t^*}\sum_{k=0}^{\infty}(\upsilon\gamma\theta)^k E_t\left\{\Lambda_{t,t+k}Y_{t+k|t}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{W}_{t+k}\right)\right\}$$

subject to the sequence of demand constraints

$$Y_{t+k|t} = \frac{1}{\alpha}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} \tag{8}$$

for $k = 0, 1, 2, ...$where $Y_{t+k|t}$ denotes output in period $t+k$ for a firm that last reset its price in period t and $\mathcal{W}_t \equiv W_t/\Gamma^t$ is the productivity-adjusted real wage (i.e. the real marginal cost).[15] Note that the discount factor $(\upsilon\gamma)^k$ captures the probability that the firm is still operative k periods ahead.

[13] Alternatively, a fraction θ of newly created firms "inherit" the price in the previous period for the good they replace. In either case I implicitly assume a transfer system which equalizes the wealth across members of the new cohort.

[14] Note that the price is common to all those firms, since they face an identical problem.

[15] The firm's demand schedule can be derived by aggregating (2) across cohorts.

The optimality condition associated with the problem above takes the form

$$\sum_{k=0}^{\infty}(\upsilon\gamma\theta)^k E_t\left\{\Lambda_{t,t+k}Y_{t+k|t}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{M}\mathcal{W}_{t+k}\right)\right\} = 0 \tag{9}$$

where $\mathcal{M} \equiv \frac{\epsilon}{\epsilon-1}$ is the optimal markup under flexible prices.

A first-order Taylor expansion of (9) around the zero inflation balanced growth path yields, after some manipulation:

$$p_t^* = \mu + (1 - \Lambda\Gamma\upsilon\gamma\theta)\sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma\theta)^k E_t\{\psi_{t+k}\} \tag{10}$$

where $\psi_t \equiv \log P_t\mathcal{W}_t$ is the (log) nominal marginal cost, $\mu \equiv \log\mathcal{M}$, and $\Lambda \equiv 1/(1+r)$ is the steady state stochastic discount factor. Throughout I maintain the assumption that $\Lambda\Gamma\upsilon\gamma \in [0,1)$, which guarantees that the firm's problem is well defined in a neighborhood of the zero inflation balanced growth path.

Letting $\mu_t \equiv p_t - \psi_t = -\log\mathcal{W}_t$ denote the average (log) price markup, and combining (7) and (10) yields the inflation equation:

$$\pi_t = \Lambda\Gamma\upsilon\gamma E_t\{\pi_{t+1}\} - \lambda(\mu_t - \mu) \tag{11}$$

where $\pi_t \equiv p_t - p_{t-1}$ denotes inflation and $\lambda \equiv (1-\theta)(1-\Lambda\Gamma\upsilon\gamma\theta)/\theta > 0$.[16]

The details of wage setting are not central to the main point of the paper. For convenience, I assume an ad-hoc wage schedule linking the productivity-adjusted real wage $\mathcal{W}_t$ to average work hours:

$$\mathcal{W}_t = \left(\frac{N_t}{\alpha}\right)^{\varphi} \tag{12}$$

where $N_t \equiv \int_0^{\alpha} N_t(i)di$ denotes aggregate work hours and α is the aggregate labor supply.

Wage schedule (12), together with the assumption of a constant gross markup $\mathcal{M}$ under flexible prices and production function (6), implies a natural (i.e. flexible price) level of output given by $Y_t^n = \Gamma^t\mathcal{Y}$, where $\mathcal{Y} \equiv \alpha\mathcal{M}^{-\frac{1}{\varphi}}$, which also corresponds to the natural level of employment. Note that as long as firms exercise their market power ($\mathcal{M} > 1$), aggregate employment under flexible prices be less than the aggregate labor supply α.

[16]Note that in the standard model with a representative consumer, $\Phi \equiv \beta$ and $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$ which correspond to the limit of the expressions for those coefficients as $\upsilon\gamma \to 1$ under the two environments, and given that $\Lambda\Gamma = \beta$ along a balanced growth path of the representative consumer economy.

Log-linearizing (12), and combining the resulting expressions with (11) we obtain a version of the New Keynesian Phillips curve

$$\pi_t = \Lambda\Gamma\upsilon\gamma E_t\{\pi_{t+1}\} + \kappa\widehat{y}_t \tag{13}$$

where $\kappa \equiv \lambda\varphi$, and $\widehat{y}_t \equiv \log(Y_t/Y_t^n)$ is the output gap. Note that, in contrast with the standard NK model, the coefficient on expected inflation is not pinned down by the consumer's discount factor. Instead it also depends on parameters affecting the life expectancy of firms (υ and γ), and the gap between the real interest rate and the growth rate along a balanced growth path (as captured by $\Lambda\Gamma$), all of which determine the effective "forward-lookingness" of inflation. In contrast with the standard model, however, and as discussed below, the interest rate along a balanced growth path (and hence Λ) is not uniquely determined by primitive parameters and will instead be related to the size of the bubble.

The fact that the natural level of output follows a deterministic trend is, of course, a consequence of the (deliberate) absence of fundamental shocks in the above framework. But it reflects another interesting property of the present model economy: bubble-driven fluctuations in economic activity cannot emerge under fully flexible prices.

2.3 Asset Markets

In addition to annuity contracts and a complete set of state-contingent securities, I assume the existence of markets for a number of specific assets, whose prices and returns must satisfy some equilibrium conditions. In particular, the yield i_t on a one-period nominally riskless bond purchased in period t must satisfy[17]

$$1 = (1 + i_t)E_t\left\{\Lambda_{t,t+1}\frac{P_t}{P_{t+1}}\right\} \tag{14}$$

thus implying that the relation $\Lambda \equiv 1/(1 + r)$ between the discount factor and the real return on the riskless nominal bond (r) will hold along a perfect foresight balanced growth path.

Stocks in individual firms trade at a price $Q_t^F(i)$, for $i \in [0, \alpha]$, which must satisfy the asset pricing equation:

$$Q_t^F(i) = D_t(i) + \upsilon\gamma E_t\left\{\Lambda_{t,t+1}Q_{t+1}^F(i)\right\} \tag{15}$$

[17] Note also that in the asset pricing equations, and from the viewpoint of an individual investor, the probability of remaining alive γ and the extra return $1/\gamma$ resulting from the annuity contract cancel each other.

where $D_t(i) \equiv Y_t(i)\left(\frac{P_t(i)}{P_t} - \mathcal{W}_t\right)$ denotes firm i's dividends, and $\upsilon\gamma$ is the probability that firm i survives into next period. Solving (15) forward under the assumption that $\lim_{k\to\infty}(\upsilon\gamma)^k E_t\left\{\Lambda_{t,t+k}Q^F_{t+k}(i)\right\} = 0$, and aggregating across firms:

$$Q^F_t \equiv \int_0^\alpha Q^F_t(i)di$$
$$= \sum_{k=0}^{\infty}(\upsilon\gamma)^k E_t\{\Lambda_{t,t+k}D_{t+k}\} \tag{16}$$

where $D_t \equiv \int_0^\alpha D_t(i)di$ denotes aggregate dividends. Note the fact that individual firms are finitely-lived makes it possible for the aggregate value of currently traded firms to be finite even if the interest rate is below the growth rate of aggregate dividends.

Much of the analysis below focuses on intrinsically worthless assets –i.e., assets generating no dividend, pecuniary or not– which may yet be traded at a positive price, constituting a pure bubble.[18] Let $Q^B_t(j)$ denote the price of one such bubble asset. In equilibrium that price must satisfy the condition

$$Q^B_t(j) = E_t\{\Lambda_{t,t+1}Q^B_{t+1}(j)\} \tag{17}$$

as well as the non-negativity constraint $Q^B_t(j) \geq 0$ (given free disposal), for all t.

Let Q^B_t denote the aggregate value of bubble assets in period t. In equilibrium, that variable evolves over time according to the following two equations:

$$Q^B_t = U_t + B_t \tag{18}$$

$$Q^B_t = E_t\{\Lambda_{t,t+1}B_{t+1}\} \tag{19}$$

where $U_t \equiv Q^B_{t|t} \geq 0$ is the value of a new bubbles introduced by cohort t at birth,[19] and $B_t \equiv \sum_{s=-\infty}^{t-1} Q^B_{t|s} \geq 0$ is the aggregate value in period t of bubble assets that were already available for trade in period $t-1$, with $Q^B_{t|s}$ denoting the period t value of bubble assets introduced in period $s \leq t$. Note that the introduction of new bubble assets by incoming cohorts makes it possible for an aggregate bubble to re-emerge after a hypothetical collapse, thus overcoming a common criticism of early rational bubble models. A similar environment with bubble creation was first introduced

[18] In Jean Tirole's words, pure bubbly assets are "best thought of as pieces of paper."

[19] Think of pieces of paper of a cohort-specific color or stamped with the birth year of their originators.

and analyzed in Martín and Ventura (2012) in the context of an overlapping generations model with financial frictions.[20]

Note that in the previous environment, the initial financial wealth of a member of a cohort born in period t is given by:

$$A_{t|t} = Q^F_{t|t} + U_t/(1-\gamma)$$

where $Q^F_{t|t}$ is the value in period t of a newly created firm.

2.4 Market Clearing

Goods market clearing requires $Y_t(i) = (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s}(i)$ for all $i \in [0,\alpha]$. Letting $Y_t \equiv \left(\alpha^{-\frac{1}{\epsilon}}\int_0^\alpha Y_t(i)^{1-\frac{1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}$ denote aggregate output, we have:

$$\begin{aligned} Y_t &= (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s} \\ &= C_t \end{aligned}$$

Note also that in equilibrium

$$\begin{aligned} N_t &= \int_0^\alpha N_t(i)di \\ &= \Delta_t^p \mathcal{Y}_t \\ &\simeq \mathcal{Y}_t \end{aligned}$$

where $\mathcal{Y}_t \equiv Y_t/\Gamma^t$ is aggregate output normalized by productivity and $\Delta_t^p \equiv \frac{1}{\alpha}\int_0^\alpha (P_t(i)/P_t)^{-\epsilon}di \simeq 1$ is an index of relative price distortions, which equals one up to a first-order approximation near a zero inflation equilibrium.

With all securities other than stocks and bubbly assets being in zero net supply, asset market clearing requires

$$(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}A_{t|s} = Q_t^F + Q_t^B$$

Next I characterize the economy's perfect foresight, zero inflation balanced growth paths.

[20] The bubble introduced by each individual can be interpreted as being attached to the stock of his firm and hence to burst whenever the firm stops operating (i.e. with probability $1-\upsilon\gamma$).

3 Balanced Growth Paths

In a perfect foresight balanced growth (henceforth, BGP) the discount factor is constant and satisfies $\Lambda = 1/(1+r)$, as implied by (14), and where r is the associated real interest rate. Note also that zero inflation requires that $\mathcal{W} = 1/\mathcal{M}$. Combined with (12) the previous condition implies that output along the BGP, is given by $Y_t^{BGP} = \Gamma^t \mathcal{Y}$, which coincides with the natural level of output, as derived above.

Next I describe how aggregate consumption is determined. Details of the derivation can be found in the Appendix.

Let $\mathcal{C}_j$ and $\mathcal{A}_j^i$ denote, respectively, consumption and financial wealth along a BGP for an individual aged j, *normalized by productivity*, with superindex $i \in \{a, r\}$ denoting his status as active or retired. The intertemporal budget constraint for a consumer born in period s and who remains active at time t, derived by solving (1) forward and evaluating it at a BGP is given by:

$$\sum_{k=0}^{\infty} (\Lambda\Gamma\gamma)^k \, \mathcal{C}_{t+k-s} = \mathcal{A}_{t-s}^a + \frac{1}{1-\Lambda\Gamma\upsilon\gamma}\left(\frac{\mathcal{W}N}{\alpha}\right)$$

where N and $\mathcal{W}$ denote aggregate hours and the wage (the latter normalized by productivity) along the BGP.

Using the fact that $\mathcal{C}_{t+k-s} = [\beta(1+r)/\Gamma]^k \mathcal{C}_{t-s}$ –as implied by (4) evaluated at the BGP – the following consumption function can be obtained for an active individual aged j:

$$\mathcal{C}_j = (1-\beta\gamma)\left[\mathcal{A}_j^a + \frac{1}{1-\Lambda\Gamma\upsilon\gamma}\left(\frac{\mathcal{W}N}{\alpha}\right)\right] \tag{20}$$

Thus, consumption for an active individual is proportional to the sum of his financial wealth, $\mathcal{A}_j^a$, and his current and future labor income (properly discounted), $\mathcal{W}N/[\alpha(1-\Lambda\Gamma\upsilon\gamma)]$.

The corresponding consumption function for a retired individual is given by:

$$\mathcal{C}_j = (1-\beta\gamma)\,\mathcal{A}_j^r \tag{21}$$

Aggregating over all individuals, imposing the asset market clearing condition $\mathcal{A} = \mathcal{Q}^F + \mathcal{Q}^B$ (with these three variables now normalized by productivity) and using the fact that $\mathcal{Q}^F = \mathcal{D}/(1-\Lambda\Gamma\upsilon\gamma)$ and $\mathcal{Y} = \mathcal{W}N + \mathcal{D}$ along a BGP (with $\mathcal{D} \equiv (1-1/\mathcal{M})\mathcal{Y}$ denoting aggregate profits normalized by productivity), we obtain:

$$\mathcal{C} = (1-\beta\gamma)\left[\mathcal{Q}^B + \frac{1}{1-\Lambda\Gamma\upsilon\gamma}\mathcal{Y}\right] \tag{22}$$

which can be interpreted as an aggregate consumption function along the BGP. Finally, goods market clearing requires that $\mathcal{C} = \mathcal{Y}$ thus implying the following equation relating the bubble-output ratio $q^B \equiv \mathcal{Q}^B/\mathcal{Y}$ and the discount factor $*$:

$$1 = (1 - \beta\gamma)\left[q^B + \frac{1}{1 - \Lambda\Gamma\upsilon\gamma}\right] \tag{23}$$

or, equivalently,

$$q^B = \frac{\gamma(\beta - \Lambda\Gamma\upsilon)}{(1 - \beta\gamma)(1 - \Lambda\Gamma\upsilon\gamma)} \tag{24}$$

Furthermore, evaluating (19) at the BGP and letting u denote the (constant) ratio between the value of new bubbles and output along a balanced growth path, we have:

$$u = \left(1 - \frac{1}{\Lambda\Gamma}\right) q^B \tag{25}$$

Next I turn to a characterization of the possible solutions to (24) and (25) satisfying $q^B \geq 0$ and $u \geq 0$.

3.1 Bubbleless Balanced Growth Paths

Consider first a "bubbleless" BGP, with $q^B = u = 0$. Imposing that condition in (24) and (25) implies

$$\Lambda\Gamma\upsilon = \beta$$

or, equivalently,

$$r = (1 + \rho)(1 + g)\upsilon - 1$$

Note that the real interest rate along a bubbleless BGP is increasing in both υ and g. The reason is that an increase in either of those variables raises desired consumption by increasing the expected stream of future income for currently active individuals. In order for the goods market to clear, an increase in the interest rate is called for.

When $\upsilon = 1$, the real interest rate is given (approximately) by the discount rate plus the growth rate, i.e. $r \simeq \rho + g$, as in the standard model with log utility (as assumed here).

Note also that an increase in the expected lifetime, as indexed by γ, does not have an independent effect on the real interest rate along the bubbleless BGP. The reason is that, when

$\Lambda\Gamma\upsilon = \beta$, such a change increases in the same proportion the present value of consumption and that of income for any given real rate, making an adjustment in the latter unnecessary.[21]

Finally, note for future reference that in the bubbleless BGP considered here the real interest rate r is lower than the growth rate g (i.e. $\Lambda\Gamma > 1$) if and only if $\upsilon < \beta$.

3.2 Bubbly Balanced Growth Paths

A BGP with a positive bubble corresponds to a solution to (24) and (25) satisfying $q^B > 0$ and $u \geq 0$. Thus, the existence of a BGP with a positive bubble, $q^B > 0$, requires that

$$\Lambda\Gamma\upsilon < \beta$$

On the other hand the non-negativity constraint on newly created bubbles $u \geq 0$ requires:

$$\Lambda\Gamma \geq 1$$

The two previous conditions are satisfied if and only if

$$\upsilon < \beta \tag{26}$$

If (26) is satisfied, there exists continuum of bubbly BGPs $\{q^B, u\}$ indexed by $r \in (\Gamma\upsilon/\beta - 1, \Gamma - 1]$. Note that the condition for the existence of bubbly BGPs corresponds to the real interest rate being less than the growth rate *in the bubbleless BGP*.

It can be easily checked that q^B is increasing in r, with $\lim_{r \to g} q^B = \frac{\gamma(\beta-\upsilon)}{(1-\beta\gamma)(1-\upsilon\gamma)} \equiv q^B_{\max}$, which establishes an upper bound to the size of the bubble-output ratio, given β,γ and υ. Note also that $\partial q^B_{\max}/\partial\upsilon < 0$, i.e. the upper bound on the size of the bubble is decreasing in υ over the range $\upsilon \in [0, \beta]$, and converges to zero as $\upsilon \to \beta$.

One particular such bubbly BGP has no new bubbles introduced by new cohorts, i.e. $u = 0$. Note that in that case

$$\Lambda\Gamma = 1$$

or, equivalently,

$$r = g$$

[21] The independence of the steady state real interest rate from γ is a consequence of the log utility specification assumed here. That property is not critical from the viewpoint of the present paper, since there are other factors (the probability of retirement, in particular), that can drive real interest rate to values consistent with the presence of bubbles.

with the implied bubble size given by $q^B = q^B_{\max}$. Along that BGP any existing bubble will be growing at the same rate as the economy. By contrast, along a bubbly BGP with bubble creation, $r < g$ implies that the size of the *aggregate* pre-existing bubble will be shrinking over time relative to the size of the economy, with newly created bubbles filling up the gap so that the size of the aggregate bubble relative to the size of the economy remains unchanged.

Summing up, one can distinguish two regions of the parameter space relevant for the possible existence of bubbly BGPs:

(i) $\beta \leq \upsilon \leq 1$. In this case, the BGP is unique and bubbleless and associated with a real interest rate given by $r = \Gamma\upsilon/\beta - 1 > g$.

(ii) $0 < \upsilon < \beta$. In this case multiple BGPs coexist. One of them is bubbleless, with $r = r_0 \equiv \Gamma\upsilon/\beta - 1 < g$. In addition, there exists a continuum of bubbly BGPs, indexed by the real interest rate $r \in (r_0, g]$, and associated with a bubble size (relative to output) $q^B \in (0, q^B_{\max}]$, given by (24).

Figure 1 summarizes graphically the two regions with their associated BGPs.

3.3 A Brief Detour: Bubbly Equilibria and Transversality Conditions

Equilibria with bubbles on assets in positive net supply can be ruled out in an economy with an infinite lived representative consumer.[22] In that economy, any positive net supply of that asset must be necessarily held by the representative consumer, implying

$$\lim_{T\to\infty} E_t \left\{\Lambda_{t,t+T} A_{t+T}\right\} \geq \lim_{T\to\infty} E_t \left\{\Lambda_{t,t+T} Q^B_{t+T}(j)\right\}$$

Given that the bubble component of any asset must satisfy

$$\lim_{T\to\infty} E_t \left\{\Lambda_{t,t+T} Q^B_{t+T}(j)\right\} = Q^B_t(j)$$

it follows that $\lim_{T\to\infty} E_t \left\{\Lambda_{t,t+T} A_{t+T}\right\} \geq Q^B_t(j)$. But the consumer's transversality condition requires that $\lim_{T\to\infty} E_t \left\{\Lambda_{t,t+T} A_{t+T}\right\} = 0$. Given that free disposal requires that $Q^B_t(j) \geq 0$, it follows that $Q^B_t(j) = 0$ for all t.

Note that the previous reasoning cannot be applied to an overlapping generations economy like the one developed above. The reason is that in that case it is no longer true that the positive

[22] See, e.g. Santos and Woodford (1997) for a discussion of the conditions under which rational bubbles can be ruled out in equilibrium.

net supply of any bubbly asset must be held (asymptotically) by any individual agent, since it can always be passed on to future cohorts (and it will in equilibrium). In fact, it is easy to check that in the model above the individual transversality condition is satisfied along any BGP, bubbly or bubbleless. As shown in the appendix, for an individual born in period $s \leq t$ it must be the case that along any BGP

$$\lim_{T \to \infty} \gamma^T E_t \left\{ \Lambda_{t,t+T} A_{t+T|s} \right\} = 0$$

implying that the transversality condition is satisfied along any admissible BGP. It is straightforward to show that this will be the case along any equilibrium that remains in a neighborhood of a BGP, of the kind analyzed below.

3.4 Plausibility of Bubbly BGPs: Some Rough Numbers

Next I discuss plausible settings for the parameters of the model involved in the above characterization of the BGPs. To calibrate γ I use the expected lifetime at age 16, which is 63.2 years in the U.S., and set $\gamma = 1 - \frac{1}{4 \cdot (62.3)} \simeq 0.996$. I use the average employment ratio (relative to population aged 16 and over), which is (roughly) 0.6 on average over the period 1960-2016 as a proxy for α. Conditional on the previous settings for α and γ one can derive $\upsilon \simeq 0.9973$. Thus, the analysis above implies that the existence of bubbly balanced growth paths requires that $\beta > 0.9973$.

Unfortunately, the latter condition cannot be verified easily since that parameter is not identified by the above model once the existence of bubbles is allowed for. This is in contrast with the standard representative agent model, for which there is a tight connection between the discount rate and the real interest rate along a BGP.[23] On the other hand, casual introspection suggests that a discount factor of about 0.9 applied to utility 10 years from today (as implied by the lower bound $\beta = 0.9973$) falls within the range of plausibility.

Alternatively, one may examine directly the relation between the average real interest rate, r, and the average growth rate of output, g, two observable variables. As discussed above, the existence of bubbly BGPs requires that $r \leq g$. Using data on 3-month Treasury bills, GDP deflator and (per capita) GDP, the average values for those variables in the U.S. over the period 1960Q1-

[23] Note that $\beta = 0.9983$ implies a discount factor of 0.9342 applied to utility 10 years from today, which seems entirely plausible.

2015Q4 are $r = 1.4\% \div 4 = 0.35\%$ and $g = 1.6\% \div 4 = 0.4\%$ (or, equivalently, $\Lambda = 0.9965$ and $\Gamma = 1.004$), values which satisfy the inequality condition necessary for the existence of bubbles. Note also that the above calibration implies $\Lambda\Gamma\upsilon\gamma \simeq 0.9939$, thus satisfying the condition for a well defined intertemporal budget constraint.

For some of the quantitative analyses below I set the discount factor to be $\beta = 0.998$. This is admittedly, an arbitrary choice, but it consistent with the existence of bubbly BGPs, with associated real interest rates given by the interval $[0.003348, 0.004]$.

4 Bubbles and Equilibrium Fluctuations

Having characterized the BGPs of the model economy, in the present section I shift the focus to the analysis of the equilibrium dynamics in a neighborhood of a given BGP. In particular, I am interested in determining the conditions under which equilibrium fluctuations unrelated to fundamental shocks may emerge, as well as the role that variations in the size of the aggregate bubble and the nature of monetary policy may play in such fluctuations.

As in the standard analysis of the NK model with a representative agent, I restrict myself to equilibria that remain in a neighborhood of a BGP, and approximate the equilibrium dynamics by means of the log-linearized equilibrium conditions.[24] I leave the analysis of the global equilibrium dynamics –including the possibility of switches between BGPs, the existence of a zero lower bound on interest rates, and other nonlinearities– to future research. Secondly, in analyzing the model's equilibrium I ignore the existence of fundamental shocks, and focus instead on the possibility of fluctuations driven by self-fulfilling expectations and on the role of bubbles as a source of those fluctuations.[25]

I start by deriving the log-linearized equilibrium conditions around a BGP. In contrast with the NK model with a representative agent, the individual consumer's Euler equation and the goods market clearing condition are no longer sufficient to derive an equilibrium relation determining aggregate output as a function of interest rates (i.e. the so-called dynamic IS equation).[26] In-

[24] See, e.g., Woodford (2003) or Galí (2015).

[25] As the analysis of the equilibrium dynamics below will make clear, in the absence of bubbles and/or multiplicity of equilibria the economy's behavior in response to fundamental shocks involves no significant differences relative to that of the standard New Keynesian model with a representative agent.

[26] Formally, one can use the definition of aggregate consumption and the individual Euler equations to derive the

stead, the derivation of such a relation requires solving for an aggregate consumption function, by aggregating the individual consumption functions obtained by combining the consumer's Euler equation and the intertemporal budget constraint. Since no exact representation exists for the individual consumption function, I derive a log-linear approximation of that function around a perfect foresight BGP. Then I aggregate the resulting individual consumption functions to obtain an (approximate) *aggregate* consumption function. See the Appendix for detailed derivations.

The resulting representation of the equilibrium dynamics takes a very simple form, involving only a few easily interpretable equations, as shown next.

Letting $\widehat{c}_t \equiv \log(C_t/\Gamma^t \mathcal{C})$ and $\widehat{y}_t \equiv \log(Y_t/\Gamma^t \mathcal{Y})$ denote log deviation of aggregate consumption and output from their value along a BGP, the goods market clearing condition can be written as:

$$\widehat{y}_t = \widehat{c}_t \tag{27}$$

As shown in the Appendix, the aggregate consumption function can be written, up to a first order approximation, as follows:

$$\widehat{c}_t = (1 - \beta\gamma)(\widehat{q}_t^B + \widehat{x}_t) \tag{28}$$

where $\widehat{q}_t^B \equiv q_t^B - q^B$ with $q_t^B \equiv Q_t^B/(\Gamma^t \mathcal{Y})$ denoting the size of the aggregate bubble normalized by trend output and where

$$\widehat{x}_t \equiv \sum_{k=0}^{\infty} (\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{y}_{t+k}\} - \frac{\Lambda\Gamma\upsilon\gamma}{1 - \Lambda\Gamma\upsilon\gamma} \sum_{k=0}^{\infty} (\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{r}_{t+k}\} \tag{29}$$

is the *fundamental* component of aggregate wealth, i.e. the expected discounted sum of current and future income), with $\widehat{r}_t = \widehat{i}_t - E_t\{\pi_{t+1}\}$ denoting the real interest rate, and $\widehat{i}_t \equiv \log[(1+i_t)/(1+r)]$ the nominal rate, all expressed in deviations from their values in the zero inflation BGP.

Note that $\widehat{x}_t$ can be conveniently rewritten in recursive form as:

$$\widehat{x}_t = \Lambda\Gamma\upsilon\gamma E_t\{\widehat{x}_{t+1}\} + \widehat{y}_t - \frac{\Lambda\Gamma\upsilon\gamma}{1 - \Lambda\Gamma\upsilon\gamma}(\widehat{i}_t - E_t\{\pi_{t+1}\}) \tag{30}$$

Log-linearization of (19) around a BGP yields the equations describing fluctuations in the aggregate bubble:

$$\widehat{q}_t^B = \Lambda\Gamma E_t\{\widehat{b}_{t+1}\} - q^B(\widehat{i}_t - E_t\{\pi_{t+1}\}) \tag{31}$$

aggregate relation:

$$E_t\{\widehat{c}_{t+1}\} = \gamma\left(\widehat{c}_t + \widehat{i}_t - E_t\{\pi_{t+1}\}\right) + (1-\gamma)\, E_t\{\widehat{c}_{t+1|t+1}\}$$

where $E_t\{\widehat{c}_{t+1|t+1}\}$ cannot be expressed immediately as a function of $\widehat{c}_t$.

$$\widehat{q}_t^B = \widehat{b}_t + \widehat{u}_t \tag{32}$$

where $\widehat{u}_t \equiv u_t - u$, with $u_t \equiv U_t/(\Gamma^t \mathcal{Y})$ denoting the size of the newly introduced bubble normalized by trend output and u its value along the BGP.

The New Keynesian Phillips curve derived above and given by

$$\pi_t = \Lambda\Gamma\upsilon\gamma E_t\{\pi_{t+1}\} + \kappa\widehat{y}_t \tag{33}$$

completes the *non-policy block* of the system of difference equations describing the model's equilibrium in a neighborhood of a BGP, where the latter is defined by a pair (q^B, Λ) satisfying the BGP conditions derived in the previous section.

In order to close the model a description of monetary policy is needed. To keep things as simple as possible, in much of the analysis below I assume an interest rate rule of the form

$$\widehat{i}_t = \phi_\pi \pi_t + \phi_q \widehat{q}_t^B \tag{34}$$

Note that the previous rule combines the usual flexible inflation targeting motive (measured by coefficient ϕ_π, and consistent with a zero inflation target) with a "leaning against the bubble" one (parameterized by ϕ_q).[27] In what follows, I assume the central bank takes as given the BGP around which the economy fluctuates (and, hence, r and q^B).

Equations (27) through (34) describe the equilibrium dynamics of the model economy in a neighborhood of a *given* BGP. A quick glance at those equations makes clear that an outcome with $\widehat{y}_t$, π_t, $\widehat{q}_t^B$, $\widehat{u}_t$ and the remaining variables all equal to zero at all times always constitutes a solution to the system of difference equations. Thus, the perfect foresight BGP itself is always an equilibrium. This should not be surprising, given that no fundamental shocks have been introduced. The question of interest, however, is whether that outcome is the only possible equilibrium and, more precisely, whether other equilibria exist involving aggregate fluctuations that are *bubble-driven*. In order to isolate bubble-driven fluctuations, however, I also need to rule out equilibria involving expectations-driven (i.e. sunspot) fluctuations unrelated to the existence of bubbles. While the conditions that allow one to rule out such fluctuations in the standard NK model are well known, it is important to check whether the introduction of finite-lives and retirement has, by itself, any consequences in that regard.

[27] Given the close link between inflation and the output gap implied by (33), there is little loss of generality (and some gains in algebraic and expository convenience) in abstracting from a term involving the output gap in the rule, as proposed in Taylor (1993).

4.1 Equilibrium Dynamics in the Absence of Bubbles

Consider equilibrium conditions (27) through (34) in the absence of bubbles, i.e. after imposing $q_t^B = u_t = 0$ for all t in the above equilibrium conditions. Combining (27), (28), and (30) we obtain a version of the dynamic IS equation for the overlapping generations economy (without bubbles):

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - (\widehat{i}_t - E_t\{\pi_{t+1}\}) \tag{35}$$

Note that the resulting dynamic IS equation takes the same form as in the standard model. In particular, we note that the presence of finite horizons *by itself* does not help overcome the "forward guidance puzzle" uncovered by Del Negro et al. (2015): the impact of an expected change in the interest rate on current output does not decline with the horizon of the policy intervention.

On the other hand, in the absence of bubbles, $\Lambda\Gamma\upsilon = \beta$, so we can write the New Keynesian Phillips curve (33) as

$$\pi_t = \beta\gamma E_t\{\pi_{t+1}\} + \kappa\widehat{y}_t \tag{36}$$

Combining the previous two equations with the interest rate rule (34), and using the latter to eliminate the interest rate in (35), yields the system

$$\begin{bmatrix} \widehat{y}_t \\ \pi_t \end{bmatrix} = \mathbf{A}_0 \begin{bmatrix} E_t\{\widehat{y}_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

where

$$\mathbf{A}_0 \equiv \Omega_0 \begin{bmatrix} 1 & 1-\beta\gamma\phi_\pi \\ \kappa & \kappa+\beta\gamma \end{bmatrix}$$

and $\Omega_0 \equiv \frac{1}{1+\kappa\phi_\pi}$. The solution $\widehat{y}_t = \pi_t = 0$ for all t is (locally) unique if and only if the two eigenvalues of $\mathbf{A}_0$ lie inside the unit circle. A necessary and sufficient condition for the latter property to be satisfied is given by $\phi_\pi > 1$, a condition identical to the one in the standard model, and often referred to as the Taylor principle. Thus, if that condition is satisfied, there is no room for expectations-driven (sunspot) fluctuations to emerge in the economy without bubbles, and given the assumption of no fundamental shocks.

In what follows, I assume that condition $\phi_\pi > 1$ is satisfied whenever (34) is in place. That assumption guarantees that any stationary fluctuations consistent with equilibria must be associated with the presence of bubbles, and is thus different in nature from the familiar sunspot fluctuations that emerge in the standard NK model as a result of a passive monetary policy rule.[28]

[28] See, e,g, Clarida et al. (2000).

4.2 Bubble-Driven Fluctuations

Nest I analyze the possibility of bubble-driven fluctuations in the OLG-NK model. Combining equations (27), (28), and (30) one can obtain the dynamic IS equation:

$$\widehat{y}_t = \Phi E_t\{\widehat{y}_{t+1}\} - \Upsilon\Phi(\widehat{i}_t - E_t\{\pi_{t+1}\}) + \frac{1-\beta\gamma}{\beta\gamma}(\widehat{q}_t^B - \Lambda\Gamma\upsilon\gamma E_t\{\widehat{q}_{t+1}^B\}) \tag{37}$$

where $\Phi \equiv \frac{\Lambda\Gamma\upsilon}{\beta} \in (0,1]$ and $\Upsilon \equiv \frac{1-\beta\gamma}{1-\Lambda\Gamma\upsilon\gamma} \in (0,1]$. The previous equation can be combined with (31) and (32) to yield

$$\widehat{y}_t = \Phi E_t\{\widehat{y}_{t+1}\} - \Psi(\widehat{i}_t - E_t\{\pi_{t+1}\}) + \Theta\widehat{q}_t^B \tag{38}$$

where $\Theta \equiv \frac{(1-\beta\gamma)(1-\upsilon\gamma)}{\beta\gamma} > 0$ and $\Psi \equiv \Upsilon\Phi\left(1 + \frac{\upsilon\gamma(1-\Phi)}{\Phi(1-\beta\gamma)}\right)$. Note that (38) has a form similar to the dynamic IS equation in a standard NK model augmented with preference shocks, with fluctuations in the bubble now playing the role of a demand shifter. In contrast with the standard model though, the coefficients on expected inflation and interest rate will generally be different from unity. In particular, the fact that $\Lambda\Gamma\upsilon < \beta$ in a bubbly BGP implies that $0 < \Phi < 1$. Accordingly, the impact on output of a given anticipated change in the interest rate declines with the horizon, thus avoiding the extreme version of the so-called "forward guidance puzzle."[29]

In addition to (38), the equilibrium dynamics around a bubbly BGP are described by the bubble equation (31), the New Keynesian Phillips curve (33), and the interest rate rule (34). Thus the description of the equilibrium dynamics can be reduced to a four equation system, including three equations that have the same interpretation as in the standard NK model, as well as a fourth equation describing the evolution of the aggregate bubble.

Using (34) to eliminate the nominal interest rate in the remaining equilibrium conditions, we can represent the resulting system of difference equations in a compact way as follows:

$$\mathbf{A}_0\mathbf{x}_t = \mathbf{A}_1 E_t\{\mathbf{x}_{t+1}\}$$

[29] The previous result can be shown to be linked to the fact that the factor $\Lambda\Gamma\upsilon\gamma$ at which future income is discounted when determining fundamental wealth (see (30)) differs from the term $\beta\gamma$ in the marginal propensity to consume in the presence of the bubble in the BGP, as implied by (23). This will also be the case in the presence of an asset other than the bubble which is in (i) positive net supply, (ii) it is not a claim to future output, and (iii) which has a positive value for current consumers (net of liabilities associated with it, e.g. taxes). An example of such an asset is government debt in the presence of finite horizons, i.e. when Ricardian equivalence does not hold. The presence of such debt is key in accounting for the muting of the "forward guidance puzzle" in Del Negro et al. (2015).

where $\mathbf{x}_t \equiv [\widehat{y}_t,\, \pi_t,\, \widehat{q}_t^B]'$ and

$$\mathbf{A}_0 \equiv \begin{bmatrix} 1 & \Psi\phi_\pi & \Psi\phi_q - \Theta \\ -\kappa & 1 & 0 \\ 0 & q^B\phi_\pi & 1 + q^B\phi_q \end{bmatrix} \quad ; \quad \mathbf{A}_1 = \begin{bmatrix} \Phi & \Psi & 0 \\ 0 & \Lambda\Gamma\upsilon\gamma & 0 \\ 0 & q^B & \Lambda\Gamma \end{bmatrix}$$

The solution to the previous dynamical system is (locally) unique if and only if the three eigenvalues of the companion matrix $\mathbf{A} \equiv \mathbf{A_0^{-1}A_1}$ lie inside the unit circle. If that condition is satisfied, the equilibrium is given by $\widehat{y}_t = \pi_t = \widehat{q}_t^B = 0$ for all t and, hence, it involves no bubble-driven fluctuations. That uniqueness condition, however, is far from being guaranteed in all cases, even for empirically plausible values of the rule coefficients. This is illustrated in Figure 2, which displays the uniqueness and indeterminacy regions on the (ϕ_π, ϕ_q) plane, under alternative assumptions regarding the real interest rate (and, hence, the bubble size) in the underlying BGP. In addition to the baseline values for β, γ, and υ introduced above, I assume $\kappa = 0.0424$ as a baseline value for the slope of the New Keynesian Phillips curve. That value is consistent with $\theta = 0.75$ and $\varphi = 0.5$. The former setting is consistent with an average duration of individual prices of 4 quarters, in accordance with much of the micro evidence. The setting for φ is consistent with (12) and the observation that the standard deviation of the (log) real wage is roughly half the size that of (log) hours worked (both HP-detrended), based on postwar U.S. data.

As shown in Figure 2, for $r = 0.00335$ –a value near the lower bound r_0 consistent with a BGP– indeterminacy is pervasive, independently of the value of ϕ_π and ϕ_q, at least for the range of those coefficients displayed in the Figure. It can be shown that, as $r \to r_0$ this is the case for *any* value of ϕ_π and ϕ_q Furthermore, if $\phi_\pi < 1$, that indeterminacy is two-dimensional with the possibility of expectations-driven bubble fluctuations added to the familiar source of indeterminacy associated with a passive monetary policy. As one considers BGPs associated with larger bubble-output ratios and higher real interest rates, we see that a region with a unique equilibrium emerges, and grows larger as r increases. When the interest rate is close enough to its upper bound ($r = g = 0.004$), the equilibrium is seen to be determinate whenever $\phi_\pi > 1$, and even for a range of values of ϕ_π less than one, as long as ϕ_q is positive.

For the sake of illustration, Figure 3 displays the simulated bubble-driven fluctuations, corresponding to a calibration of the OLG-NK model consistent with one-dimensional indeterminacy. In particular I assume $r = 0.0035$ (the estimated average interest rate in postwar U.S.), $\phi_\pi = 1.5$ (the value proposed in Taylor (1993) and often used in related exercises) and $\phi_q = 0$ (a likely relevant value for the Fed and other central banks, at least before the financial crisis). Note that in both the standard NK model and in the OLG-NK model *without bubbles*, the fact that $\phi_\pi > 1$ would guarantee a unique equilibrium. By contrast, once bubbles are allowed for, that rule is seen to be consistent with highly persistent fluctuations in output, inflation and the bubble. The intuition behind those fluctuations should be clear given the equilibrium relations discussed above: if, starting from a BGP, individuals become "optimistic" and expect bubble assets to command a higher price in the future they will be willing to pay a higher price for those assets today; the resulting increase in financial wealth will raise aggregate demand, which firms will be willing to meet by increasing output. The consequent reduction in average markups will lead firms which can adjust their prices to raise them, thus generating positive inflation. The endogenous policy response to the higher inflation partly dampens the increase in aggregate demand, but not sufficiently to fully stabilize it.

How effective is a monetary policy that leans against the bubble, as captured by a positive value for ϕ_q, at preventing the possibility of bubble-driven fluctuations? Figure 4 seeks to answer that question by showing the threshold value for ϕ_q above which stationary bubble-driven fluctuations can be ruled out, given $\phi_\pi = 1.5$. As the figure makes clear, the extent to which a systematic interest rate response to the aggregate bubble may rule out such fluctuations depends very much on the steady state real interest rate (and, hence on the size of the bubble-output ratio along the corresponding BGP). As discussed above, if the interest rate is close enough to the growth rate (i.e. bubble-output ratio is sufficiently large), a policy that focuses on stabilizing inflation is sufficient to prevent bubble-driven fluctuations, even with $\phi_q = 0$. To see why this is the case it is useful to combine (31) and (32) to yield:

$$\widehat{q}_t^B = (\Lambda\Gamma)^{-1}\widehat{q}_{t-1}^B + q^B\widehat{r}_{t-1} + \varepsilon_t$$

where $\varepsilon_t \equiv b_t - E_{t-1}\{b_t\} + u_t$ denotes the aggregate bubble innovation. When the bubble-output ratio q^B is large and the interest rate is close to the growth rate ($\Lambda\Gamma \lesssim 1$) a policy that involves even a small response of the real rate to a larger bubble will trigger an explosive path for the latter, inconsistent with an equilibrium. On the other hand, as q^B approaches zero and $(\Lambda\Gamma)^{-1}$ diverges from unity, it takes an ever more aggressive interest rate response in order to rule out the possibility of stationary fluctuations in the aggregate bubble (and, hence, in output and inflation).

This is captured in Figure 4 which shows how the threshold value for ϕ_q rises quickly as r declines, and becomes infinite as that variable approaches 0.0033, its lower bound.

4.2.1 A Particular Case: Bubble-Driven Fluctuations around the Bubbleless BGP

The previous analysis has focused on fluctuations in a neighborhood of a perfect foresight bubbly BGP. But given the existence of a continuum of such BGPs one may wonder why an economy would tend to gravitate towards the same one. From that viewpoint, the possibility of bubble-driven fluctuations near a *bubbleless* BGP may be viewed as particularly appealing since that BGP would seem to be a natural "attractor" in the face of a bursting aggregate bubble. The present section focuses on that particular case.

Note that in a neighborhood of the bubbleless BGP the aggregate bubble q_t^B must now satisfy the equilibrium conditions

$$q_t^B = (\beta/\upsilon)E_t\{b_{t+1}\} \tag{39}$$

$$q_t^B = b_t + u_t \tag{40}$$

where $(q_t^B, u_t) \geq 0$ for all t. Note that the assumption $q^B = 0$ made here implies that interest rate changes do not affect (up to a first order approximation) the expected evolution of the bubble, which simplifies the analysis considerably. Furthermore, under the maintained assumptions that $\upsilon < \beta$ (implying $r < g$) and $0 < u_t < \overline{u}$ for all t, it is easy to check that $\lim_{k\rightarrow\infty} E_t\{q_{t+k}^B\} < \frac{\overline{u}}{1-\upsilon/\beta}$, i.e. the bubble is non-explosive and its average long run value can be made arbitrarily close to zero.

The previous environment allows for the possibility of *recurrent* booms and busts driven by an exogenous stochastic bubble of the kind proposed in Blanchard (1979). To illustrate that possibility, consider a bubble that evolves according to the process.

$$q_t^B = \begin{cases} \frac{\upsilon}{\beta\delta} q_{t-1}^B + u_t & \text{with probability } \delta \\ u_t & \text{with probability } 1-\delta \end{cases} \tag{41}$$

where $\{u_t\}$ follows a martingale-difference process with positive support and constant mean $\overline{u} \gtrsim 0$. It is easy to check that the previous process satisfies (39) and (40), as well as the non-negativity condition. Note that the aggregate bubble will display persistent fluctuations over time, experiencing recurrent (though unpredictable) collapses, before being rekindled again by newly introduced bubbles.

The previous stochastic bubble process can be combined with (33), (34), and (37), all evaluated at the perfect foresight bubbleless BGP (i.e. with $\Lambda\Gamma\upsilon = \beta$ and $q^B = 0$) to describe the equilibrium dynamics. In the special case considered here, the equilibrium output gap and inflation can be solved for in closed form, as a function of the current value of the aggregate bubble:

$$\widehat{y}_t = (1 - \beta\gamma)\Omega(\Theta - \phi_q)q_t^B$$

$$\pi_t = \kappa\Omega(\Theta - \phi_q)q_t^B$$

where $\Omega \equiv 1/[(1 - \beta\gamma)(1 - \upsilon/\beta) + \kappa(\phi_\pi - \upsilon/\beta)] > 0$.

Figure 5 displays a simulated equilibrium path generated by the previous model. In addition to the baseline parameter settings introduced above, I set $\delta = 0.95$ as a value describing the process for the bubble and draw $\{u_t\}$ from a uniform distribution on $[0, 0.001]$. As above I assume $\phi_\pi = 1.5$ and $\phi_q = 0$. Note that the simulated fluctuations display a characteristic "boom-bust" pattern often found in accounts of historical bubble episodes, with the output gap and inflation moving in step with the bubble.

5 Bubbles and Monetary Policy Design

In the present section I discuss the implications of alternative monetary policies in the OLG-NK economy developed above. For concreteness, I adopt the perspective of a central bank that seeks to stabilize inflation.[30] Note that the lack of a trade-off between inflation and output gap implied by (33) makes an inflation targeting policy also consistent with output gap stabilization. Next I discuss three candidate strategies to implement that inflation targeting policy.

A first strategy, implicit in the analysis above, consists in adopting a rule that would render bubble-driven fluctuations inconsistent with equilibrium. As discussed in section 4, a sufficiently aggressive "leaning against the bubble" policy, implemented by means of an interest rate rule like (34) with a large setting for ϕ_q may in some instances serve that purpose. An important limitation of such a strategy is that if the economy is too close to the bubbleless BGP, there may not be a finite value of ϕ_q that would make that strategy feasible, or the required value for ϕ_q may be far too large, which would render it unfeasible in practice.

[30] The presence of uninsurable cohort-specific shocks linked to new bubbles renders the analysis of the fully optimal monetary policy hard to tackle. I leave its exploration for future research.

Rather than ruling out equilibria involving bubble fluctuations, an alternative policy strategy would instead seek to offset the effects of those fluctuations on aggregate demand, through a suitable adjustment in the interest rate. Given (38), that policy can be implemented with the rule

$$\widehat{i}_t = \phi_\pi \pi_t + (\Theta/\Psi)\widehat{q}_t^B \tag{42}$$

where $\phi_\pi > 1$. Combining (42) with (38) and (33), one can check that the equilibrium is locally unique and given by $\widehat{y}_t = \pi_t = 0$ for all t. In other words, that policy would succeed in fully insulating aggregate output and inflation from fluctuations in the bubble. Note, however, that such a policy would *not* prevent fluctuations in the bubble. Instead, the latter would evolve in equilibrium according to:

$$\widehat{q}_t^B = \chi \widehat{q}_{t-1}^B + \varepsilon_t \tag{43}$$

where $\varepsilon_t \equiv b_t - E_{t-1}\{b_t\} + \widehat{u}_t$ and $\chi \equiv \frac{1}{\Lambda\Gamma}\left(1 + \frac{q^B\Theta}{\Psi}\right)$, with the latter being less than one under the conditions that allow for non-explosive bubble-driven equilibria, as discussed above. Note that this in turn requires that q^B is sufficiently small.[31] As long as that condition is satisfied, the aggregate bubble will display non-explosive fluctuations around its BGP value, as described by (43), but those fluctuations will not translate into output or inflation fluctuations, for the response of the nominal (and real) interest rate will exactly offset their impact on aggregate demand.

The same outcome, however, could be approximated if the central bank were to adopt a rule that directly targeted inflation and/or the output gap, while ignoring the existence of a bubble and its eventual fluctuations. If that rule were indeed to succeed in stabilizing inflation, then it would have to be the case that, in equilibrium, $\widehat{i}_t = (\Theta/\Psi)\widehat{q}_t^B$, as implied by (38). An example of such a rule is given by $\widehat{i}_t = \phi_\pi \pi_t$ with ϕ_π arbitrarily large. The main advantage of such a strategy, relative to the first one is that it is always effective, even in the limiting case when fluctuations emerge near the bubbleless BGP. Relative to the second strategy, a direct inflation targeting rule does not require that the bubble be observed nor knowledge of the model's parameters.

Figure 6 illustrates the previous point by displaying the standard deviation of the output gap as a function of ϕ_π and ϕ_q, given $r = 0.0035$, and keeping the remaining parameter settings unchanged. The figure points to the potential destabilizing effects of a "leaning against the bubble"

[31] Note that $\lim_{q^B \to 0} \chi = \upsilon/\beta$ which is less than one under the conditions that guarantee the existence of bubbles.

policy that is not "surgically" calibrated, and which stands in contrast to the greater robustness of inflation targeting policies, which can approximate the desired outcome arbitrarily well.

Figure 7 displays the standard deviation of the bubble, as a function of ϕ_π and ϕ_q, and under an identical calibration of the remaining parameters. The figure illustrates a somewhat surprising aspect of "leaning against the bubble" policies: contrary to conventional wisdom, such policies may end up increasing the size of bubble fluctuations. That outcome, already uncovered in Galí (2014) in the context of a two-period monetary OLG model, follows from the requirement that any existing bubble grows (in expectation) at the rate of interest. Thus, if the latter increases with the bubble, the size of bubble fluctuations may be amplified. To the extent that those fluctuations are deemed undesirable –beyond their effects on the output gap and inflation– the previous finding provides an additional argument against policies that directly (and aggressively) respond to fluctuations in the bubble.

6 Concluding Comments

The NK model remains the workhorse framework for monetary policy analysis, even though it is unsuitable –in its standard formulation– to accommodate the existence of asset price bubbles and hence to address one of the key questions facing policy makers, namely, how monetary policy should respond to those bubbles. That shortcoming, however, is not tied to any key ingredient of the model (e.g. staggered price setting), but to the convenient (albeit unrealistic) assumption of an infinite-lived representative consumer. In the present paper I have developed an extension of the basic NK model featuring overlapping generations, finite lives and (stochastic) transitions to inactivity. In contrast with the standard model, the OLG-NK framework allows for the existence of rational expectations equilibria with asset price bubbles. In particular, plausible calibrations of the model's parameters are consistent with the existence of a continuum of bubbly balanced growth paths, as well as a bubbleless one (which always exists). When combined with sticky prices, fluctuations in the size of the aggregate bubble, possibly unrelated to changes in fundamentals, are a potential source of fluctuations in output and inflation.

The analysis of the properties of the model has yielded several insights regarding its implications for monetary policy.

Firstly, when one abstracts from the possibility of bubbly equilibria, the introduction of an overlapping generations structure does not change any of the qualitative properties of the standard NK model. In particular, the Taylor principle still defines the condition under which a simple interest rate rule guarantees equilibrium uniqueness. Furthermore, the presence of finite horizons *by itself* does not help overcome the so called "forward guidance puzzle" uncovered by a number of authors in the context of an infinitely-lived representative agent model.

Secondly, a "leaning against the bubble" interest rate policy, if precisely calibrated, may succeed in insulating output and inflation from aggregate bubble fluctuations. However, mismeasurement of the bubble or an inaccurate "calibration" of the policy response to bubble fluctuations may end up destabilizing output and inflation. On the other hand, and contrary to conventional wisdom, a "leaning against the bubble" policy, even when precisely implemented, does not guarantee the elimination of the bubble or the dampening of its fluctuations. In fact, under some conditions such a policy may end up *increasing* the volatility and persistence of bubble fluctuations. By way of contrast, a policy that targets inflation directly may attain the same stabilization objectives without the risks associated with a "direct" response to the bubble.

Four additional remarks, pointing to possible future research avenues are in order. Firstly, the analysis of the equilibrium dynamics above has assumed "rationality" of asset price bubbles. That assumption underlies the equilibrium conditions that individual and aggregate bubbles must satisfy, i.e. (17) and (19), respectively, and the implied log-linear approximation (31). However, the remaining equilibrium conditions, including the modified dynamic IS equations, *are still valid* even if the process followed by the aggregate bubble were to deviate from that of a *rational* bubble. That observation opens the door to analyses of the macroeconomic effects and policy implications of *alternative* specifications of the aggregate assets misvaluations.

Secondly, the analysis above suggests that, in the absence of other frictions, the rise and fall of an aggregate bubble is likely to have small quantitative effects on aggregate demand and, hence, on output and inflation. To the extent that the historical evidence points to a larger macro impact of bubbles, there may be additional channels through which that impact operates. The work of Farhi and Tirole (2011), Martin and Ventura (2012), and Asriyan et al. (2016), emphasizing the interaction of the bubble with financial frictions, points to a possible source of larger quantitative effects that one might be able to incorporate in the above framework.

Thirdly, balanced growth paths characterized by a larger bubble size are associated with a higher real interest rate. Thus, the presence of a bubble along a BGP should make it less likely for the zero lower bound on the nominal interest rate to become binding, ceteris paribus and conditional on the bubble not bursting. Similarly, the bursting of the bubble would bring along a reduction in the natural rate of interest that could pull the interest rate toward the zero lower bound. The analysis of the interaction of bubble dynamics with the zero lower bound seems an avenue worth exploring in future research.

Finally, the analyses of the equilibrium dynamics above has assumed that the central bank takes as given the BGP on which the economy settles and, hence, its associated real interest rate. That assumption is reflected in the implicit exogeneity of the real interest rate term embedded in the policy rule through the term $\widehat{i}_t \simeq i_t - r$. But while that assumption is a natural one in the context of economies whose real interest rate along a BGP is uniquely pinned down (as it is the case in the standard New Keynesian model with a representative consumer), it is no longer so in an economy like the one analyzed above, in which a multiplicity of real interest rates may be consistent with a perfect foresight BGP. Future research should explore the implications of relaxing the assumption of a policy invariant steady state real interest rate.

References

Allen, Franklin, Gadi Barlevy and Douglas Gale (2017): "On Interest Rate Policy and Asset Bubbles," mimeo.

Asriyan, Vladimir, Luca Fornaro, Alberto Martín and Jaume Ventura (2016): "Monetary Policy for a Bubbly World," mimeo.

Bernanke, Ben S. (2002): "Asset Price Bubbles and Monetary Policy," speech before the New York Chapter of the National Association of Business Economists.

Bernanke, Ben S. and Mark Gertler (1999): "Monetary Policy and Asset Price Volatility," in *New Challenges for Monetary Policy*, Federal Reserve Bank of Kansas City, 77-128.

Bernanke, Ben S. and Mark Gertler (2001): "Should Central Banks Respond to Movements in Asset Prices?" *American Economic Review* 91(2), 253-257.

Blanchard, Olivier J. (1979): "Speculative Bubbles, Crashes and Rational Expectations," *Economic Letters* 3, 387-389.

Blanchard, Olivier J. (1985): "Debt, Deficits, and Finite Horizons." *Journal of Political Economy*, 93(2), 223-247.

Borio, Claudio and Philip Lowe (2002): "Asset Prices, Financial and Monetary Stability: Exploring the Nexus," BIS Working Papers no. 114.

Carvalho, Carlos, Andrea Ferrero and Fernanda Nechio (2016): "Demographics and real interest rates: Inspecting the Mechanism," *European Economic Review*, forthcoming.

Cecchetti, Stephen G., Hans Gensberg, John Lipsky and Sushil Wadhwani (2000): *Asset Prices and Central Bank Policy*, Geneva Reports on the World Economy 2, CEPR.

Del Negro, Marco, Marc Giannoni, and Christina Patterson (2015): "The Forward Guidance Puzzle," unpublished manuscript.

Dong, Feng, Jianjun Miao, Pengfei Wang (2017): "Asset Bubbles and Monetary Policy," mimeo.

Farhi, Emmanuel and Jean Tirole (2011): "Bubbly Liquidity." *Review of Economic Studies* 79(2): 678-706.

Galí, Jordi (2014): "Monetary Policy and Rational Asset Price Bubbles," *American Economic Review*, vol. 104(3), 721-752

Gertler, Mark (1999): "Government Debt and Social Security in a Life-Cycle Economy," *Carnegie-Rochester Conference Series on Public Policy* 50, 61-110.

Kohn, Donald L. (2006): "Monetary Policy and Asset Prices," speech at an ECB colloquium on "Monetary Policy: A Journey from Theory to Practice," held in honor of Otmar Issing.

Kohn, Donald L. (2008): "Monetary Policy and Asset Prices Revisited," speech delivered at the Caton Institute's 26th Annual Monetary Policy Conference, Washington, D.C.

Martín, Alberto, and Jaume Ventura (2012): "Economic Growth with Bubbles," *American Economic Review*, vol.102(6), 3033-3058.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson (2016). "The Power of Forward Guidance Revisited," *American Economic Review* 106 (10): 3133-3158.

Nisticò, Salvatore (2012): "Monetary Policy and Stock-Price Dynamics in a DSGE Framework," *Journal of Macroeconomics* 34, 126-146.

Piergallini, Alessandro (2006): "Real Balance Effects and Monetary Policy," *Economic Inquiry* 44(3), 497-511.

Samuelson, Paul A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy* 66, 467-482.

Santos, Manuel S. and Michael Woodford (1997): "Rational Asset Pricing Bubbles," *Econometrica* 65(1), 19-57.

Schularick, Moritz and Alan M. Taylor (2012): "Credit Booms Gone Bust: Monetary Policy, Leverage Cycles and Financial Crises, 1870-2008" *American Economic Review*, 102(2), 1029-1061.

Taylor, John (2014): "The Role of Policy in the Great Recession and the Weak Recovery," *American Economic Review* 104(5), 61-66.

Tirole, Jean (1985): "Asset Bubbles and Overlapping Generations," *Econometrica* 53, 1499-1528.

Yaari, Menahem E. (1965): "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," The *Review of Economic Studies*, 32(2), 137-150

TECHNICAL APPENDIX

1. Transversality condition in a Bubbly BGP

The consumption function for an active individual born in period s is:

$$C^a_{t+T|s} = (1-\beta\gamma)\left[A^a_{t+T|s} + \frac{W_{t+T}N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}\right]$$

In particular:

$$C^a_{t|s} = (1-\beta\gamma)\left[A^a_{t|s} + \frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}\right]$$

In addition, $C_{t+T|s} = (\beta/\Lambda)^T C_{t|s}$ thus implying

$$A^a_{t+T|s} = (\beta/\Lambda)^T\left[A^a_{t|s} + (1-(\Lambda\Gamma/\beta)^T)\frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}\right]$$

On the other hand, for a retired individual

$$C^r_{t|s} = (1-\beta\gamma)A^r_{t|s}$$

Using the fact that $C^r_{t|s} = C^a_{t|s}$ we have:

$$A^r_{t|s} = A^a_{t|s} + \frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}$$

The transversality condition for an active individual takes the form:

$$\begin{aligned}
\lim_{T\to\infty}(\gamma\Lambda)^T E_t\{A_{t+T|s}\} &= \lim_{T\to\infty}(\gamma\Lambda)^T[\upsilon^T A^a_{t+T|s} + (1-\upsilon^T)A^r_{t+T|s}] \\
&= \lim_{T\to\infty}(\gamma\Lambda)^T[A^a_{t+T|s} + (1-\upsilon^T)\Gamma^T\frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}] \\
&= \lim_{T\to\infty}(\beta\gamma)^T\left[A^a_{t|s} + (1-(\Lambda\Gamma/\beta)^T)\frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}\right] + \lim_{T\to\infty}(\gamma\Lambda\Gamma)^T(1-\upsilon^T)\frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma} \\
&= \frac{W_t N/\alpha}{1-\Lambda\Gamma\upsilon\gamma}\lim_{T\to\infty}[(\beta\gamma)^T - (\Lambda\Gamma\gamma)^T + (\Lambda\Gamma\gamma)^T - (\Lambda\Gamma\upsilon\gamma)^T] \\
&= 0
\end{aligned}$$

where the maintained assumption $\Lambda\Gamma\upsilon\gamma < 1$ has been used.

2. Log-linearized individual intertemporal budget constraint

The intertemporal budget constraint as of period t for an individual born in period s and still active in period $t \geq s$ can be derived by iterating (3) forward to yield:

$$\sum_{k=0}^{\infty} \gamma^k E_t\{\Lambda_{t,t+k} C_{t+k|s}\} = A^a_{t|s} + \frac{1}{\alpha} \sum_{k=0}^{\infty} (\gamma \upsilon)^k E_t\{\Lambda_{t,t+k} W_{t+k} N_{t+k}\} \tag{44}$$

For retired individuals, the corresponding constraint is:

$$\sum_{k=0}^{\infty} \gamma^k E_t\{\Lambda_{t,t+k} C_{t+k|s}\} = A^r_{t|s} \tag{45}$$

Letting lowercase letters with a "ˆ" symbol denote the log deviation of a variable from its value along a perfect foresight balanced growth path (BGP), the left hand side term of (44) and (45) can be approximated in a neighborhood of the BGP as:

$$\begin{aligned}\sum_{k=0}^{\infty} \gamma^k E_t\{\Lambda_{t,t+k} C_{t+k|s}\} &\simeq \frac{\Gamma^t \mathcal{C}_{t-s}}{1-\beta\gamma} + \Gamma^t \sum_{k=0}^{\infty} (\Lambda \Gamma \gamma)^k \mathcal{C}_{t+k-s} E_t\{\widehat{c}_{t+k|s} + \widehat{\lambda}_{t,t+k}\} \\ &= \frac{\Gamma^t \mathcal{C}_{t-s}}{1-\beta\gamma} + \frac{\Gamma^t \mathcal{C}_{t-s}}{1-\beta\gamma} \widehat{c}_{t|s}\end{aligned}$$

where $\mathcal{C}_j$ denotes consumption of an individual aged j, normalized by current productivity, along a BGP, $\widehat{\lambda}_{t,t+k} = \log(\Lambda_{t,t+k}/\Lambda^k)$, and where I have made use of the fact that $\mathcal{C}_{t+k-s} = [\beta(1+r)/\Gamma]^k \mathcal{C}_{t-s}$ and $E_t\{\widehat{\lambda}_{t,t+k} + \widehat{c}_{t+k|s}\} = \widehat{c}_{t|s}$ (resulting from (4)).

The second term on the right hand side of (44), which is relevant only for active individuals, can be approximated around the BGP as:

$$\frac{1}{\alpha} \sum_{k=0}^{\infty} (\upsilon\gamma)^k E_t\{\Lambda_{t,t+k} W_{t+k} N_{t+k}\} \simeq \frac{\Gamma^t \mathcal{W} N}{\alpha(1-\Lambda\Gamma\upsilon\gamma)} + \left(\frac{\Gamma^t \mathcal{W} N}{\alpha}\right) \sum_{k=0}^{\infty} (\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{w}_{t+k} + \widehat{n}_{t+k} + \widehat{\lambda}_{t,t+k}\}$$

where $\mathcal{W}$ is the wage along the BGP, normalized by productivity.

3. Aggregation and derivation of log-linearized aggregate consumption Euler equation

Let $\mathcal{C}$ denote aggregate consumption (normalized by current productivity), along the BGP. One can derive the approximate relations $\mathcal{C}\widehat{c}_t = (1-\gamma)\sum_{s=-\infty}^{t} \gamma^{t-s} \mathcal{C}_{t-s} \widehat{c}_{t|s}$ and the BGP relation $\mathcal{C} = (1-\gamma)\sum_{s=-\infty}^{t} \gamma^{t-s} \mathcal{C}_{t-s}$. Those relations can be used to aggregate the log-linearized individual consumption functions across all individuals to yield:

$$\frac{\Gamma^t \mathcal{C}}{1-\beta\gamma} + \frac{\Gamma^t \mathcal{C}}{1-\beta\gamma} \widehat{c}_t = (Q^F_t + Q^B_t) + \frac{\Gamma^t \mathcal{W} N}{1-\Lambda\Gamma\upsilon\gamma} + \Gamma^t \mathcal{W} N \sum_{k=0}^{\infty} (\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{w}_{t+k} + \widehat{n}_{t+k} + \widehat{\lambda}_{t,t+k}\}$$

Note also that in a neighborhood of the BGP,

$$\begin{aligned} Q_t^F &= \sum_{k=0}^{\infty}(\upsilon\gamma)^k E_t\{\Lambda_{t,t+k}D_{t+k}\} \\ &\simeq \frac{\Gamma^t\mathcal{D}}{1-\Lambda\Gamma\upsilon\gamma} + \Gamma^t\mathcal{D}\sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{d}_{t+k} + \widehat{\lambda}_{t,t+k}\} \end{aligned}$$

where $\mathcal{D}$ denotes aggregate dividends (normalized by productivity), along a BGP.

Thus, using the approximation $\widehat{y}_t = (\mathcal{W}N/\mathcal{Y})(\widehat{w}_t + \widehat{n}_t) + (\mathcal{D}/\mathcal{Y})\widehat{d}_t$, the BGP relation $\frac{\Gamma^t\mathcal{C}}{1-\beta\gamma} = Q^B + \frac{\Gamma^t(\mathcal{W}N+\mathcal{D})}{1-\Lambda\Gamma\upsilon\gamma}$, and the goods market clearing condition $\mathcal{Y} = \mathcal{C}$, we obtain the log-linearized aggregate consumption function

$$\begin{aligned} \widehat{c}_t &= (1-\beta\gamma)\left[\widehat{q}_t^B + \sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{y}_{t+k} + \widehat{\lambda}_{t,t+k}\}\right] \\ &= (1-\beta\gamma)\left[\widehat{q}_t^B + \sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{y}_{t+k}\} - \sum_{k=1}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k \sum_{j=0}^{k-1} E_t\{\widehat{r}_{t+j}\}\right] \\ &= (1-\beta\gamma)\left[\widehat{q}_t^B + \sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{y}_{t+k}\} - \frac{\Lambda\Gamma\upsilon\gamma}{1-\Lambda\Gamma\upsilon\gamma}\sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{r}_{t+k}\}\right] \end{aligned}$$

where $\widehat{q}_t^B \equiv q_t^B - q^B$, with $q_t^B \equiv Q_t^B/(\Gamma^t\mathcal{Y})$ and q^B its corresponding value along a BGP, and $\widehat{r}_t \equiv \widehat{i}_t - E_t\{\pi_{t+1}\}$.

Note that we can write the consumption function more compactly as:

$$\widehat{c}_t = (1-\beta\gamma)(\widehat{q}_t^B + \widehat{x}_t) \tag{46}$$

where $\widehat{x}_t \equiv \sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{y}_{t+k}\} - \frac{\Lambda\Gamma\upsilon\gamma}{1-\Lambda\Gamma\upsilon\gamma}\sum_{k=0}^{\infty}(\Lambda\Gamma\upsilon\gamma)^k E_t\{\widehat{r}_{t+k}\}$ is the "fundamental" component of wealth, expressed in log deviations from its value along a BGP. It satisfies the recursive equation:

$$\widehat{x}_t = \Lambda\Gamma\upsilon\gamma E_t\{\widehat{x}_{t+1}\} + \widehat{y}_t - \frac{\Lambda\Gamma\upsilon\gamma}{1-\Lambda\Gamma\upsilon\gamma}\widehat{r}_t$$

Furthermore, log-linearization of (**??**) about a BGP yields:

$$\widehat{q}_t^B = \Lambda\Gamma E_t\{\widehat{q}_{t+1}^B\} - q^B\widehat{r}_t \tag{47}$$

In a neighborhood of the *bubbleless BGP* $q^B = 0$ and $\Lambda\Gamma\upsilon = \beta$, thus implying the approximate aggregate Euler equation:

$$\widehat{c}_t = \beta\gamma E_t\{\widehat{c}_{t+1}\} + (1-\beta\gamma)(1-\upsilon\gamma)q_t^B + (1-\beta\gamma)\widehat{y}_t - \beta\gamma\widehat{r}_t$$

Imposing goods market clearing ($\widehat{c}_t = \widehat{y}_t$) and rearranging terms, we obtain a dynamic IS equation:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} + \Phi q_t^B - \widehat{r}_t$$

where $\Phi \equiv (1-\beta\gamma)(1-\upsilon\gamma)/\beta\gamma > 0$ and $\{q_t^B\}$ must satisfy $q_t^B = (\beta/\upsilon)E_t\{q_{t+1}^B\}$ and $q_t^B \geq 0$ for all t.

Note that in a bubbleless equilibrium around the bubbleless BGP:

$$\widehat{y}_t = E_t\{\widehat{y}_{t+1}\} - \widehat{r}_t$$

which is identical to the standard dynamic IS equation of the representative consumer model.

In any *bubbly steady state*, $q^B = \frac{1}{1-\beta\gamma} - \frac{1}{1-\Lambda\Gamma\upsilon\gamma}$. Combining (46) and (47) yields the Euler equation:

$$\widehat{c}_t = \Lambda\Gamma\upsilon\gamma E_t\{\widehat{c}_{t+1}\} + (1-\beta\gamma)(1-\upsilon\gamma)\widehat{q}_t^B + (1-\beta\gamma)\widehat{y}_t - \Upsilon\upsilon\gamma\widehat{r}_t$$

where $\Upsilon \equiv \left(1 + \frac{(1-\beta\gamma)(\Lambda\Gamma-1)}{1-\Lambda\Gamma\upsilon\gamma}\right) > 1$ and $\Lambda \in \left(1, \frac{\beta}{\Gamma\upsilon}\right)$. Imposing goods market clearing:

$$\widehat{y}_t = \frac{\Lambda\Gamma\upsilon}{\beta}E_t\{\widehat{y}_{t+1}\} + \Phi\widehat{q}_t^B - \frac{\Upsilon\upsilon}{\beta}\widehat{r}_t \qquad (48)$$

In the particular case of $\delta = 0$, we have $\Lambda\Gamma = \Upsilon = 1$ thus implying a dynamic IS equation of the form:

$$\widehat{y}_t = \frac{\upsilon}{\beta}E_t\{\widehat{y}_{t+1}\} + \Phi\widehat{q}_t^B - \frac{\upsilon}{\beta}\widehat{r}_t$$

4. Conditions for equilibrium uniqueness around the bubbleless BGP.

The necessary and sufficient conditions for $\mathbf{A}_T$ to have two eigenvalues within the unit circle are: (i) $|\det(\mathbf{A}_T)| < 1$ and (ii) $|tr(\mathbf{A}_T)| < 1 + \det(\mathbf{A}_T)$ (LaSalle (1986)). Note that in the case of a bubbleless BGP, $\det(\mathbf{A}_T) = \frac{\Phi}{1+\kappa\phi_\pi} > 0$ while $tr(\mathbf{A}_0) = \frac{1+\kappa+\Phi}{1+\kappa\phi_\pi} > 0$. Condition (i) requires $\phi_\pi > (\Phi-1)/\kappa$. Condition (ii) corresponds to $\phi_\pi > 1$.

5. Sunspot Fluctuations

Let the equilibrium be described by the system of difference equations

$$\mathbf{x}_t = \mathbf{A} E_t\{\mathbf{x}_{t+1}\}$$

where $\mathbf{x}_t \equiv [\widehat{y}_t,\ \pi_t,\ \widehat{q}_t^B]'$ are all non-predetermined variables. Let $\mathbf{A}$ have $q \leq 3$ eigenvalues with modulus less than one.

Consider the transformation $\mathbf{x}_t = \mathbf{Q}\mathbf{v}_t$ where $\mathbf{QJQ}^{-1} = \mathbf{A}$ where $\mathbf{J}$ is the canonical Jordan matrix and $\mathbf{Q} \equiv [\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{q}^{(3)}]$ is the matrix of generalized eigenvectors, corresponding to the three eigenvalues. Thus,

$$\mathbf{v}_t = \mathbf{J} E_t\{\mathbf{v}_{t+1}\}$$

where $\mathbf{J} = \begin{bmatrix} \mathbf{J}_u & 0 \\ 0 & \mathbf{J}_s \end{bmatrix}$ and $\mathbf{v}_t = [\mathbf{v}_t^u,\ \mathbf{v}_t^s]'$.

Consider the case where $\mathbf{A}$ has two eigenvalues with modulus less than one and one with modulus greater than one. For concreteness, assume $|\lambda_1| < 1$, $|\lambda_2| < 1$, and $|\lambda_3| > 1$. If all eigenvalues are real $\mathbf{J}_u = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\mathbf{J}_s = \lambda_3$, and $\mathbf{q}^{(k)}$ corresponds to the eigenvector associated with eigenvalue k, for $k = 1, 2, 3$. Otherwise, if $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ are complex conjugates, $\mathbf{J}_u = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $\mathbf{J}_s = \lambda_3$ and $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are, respectively, the imaginary and real components of the eigenvector associated with the complex eigenvalues.

Thus, $\mathbf{v}_t^u = 0$ all t. In addition $\mathbf{v}_t^s = \lambda_3 E_t\{\mathbf{v}_{t+1}^s\}$, which has a stable solution:

$$\mathbf{v}_t^s = \lambda_3^{-1} \mathbf{v}_{t-1}^s + \xi_t$$

where ξ_t is a martingale-difference (univariate) process.

Thus, we have $\mathbf{x}_t = \mathbf{q}^{(3)} \mathbf{v}_t^s$ is the sunspot solution. The three variables in $\mathbf{x}_t$ will be perfectly correlated (positively or negatively), and will display a first-order autocorrelation λ_3^{-1}. In the simulations reported normalize the the third element of $\mathbf{q}^{(3)}$ to unity, in which case the innovation in $\mathbf{v}_t^s$ (denoted by ξ_t) corresonds to the innovatin in the bubble.

Consider next the case where $\mathbf{A}$ has one eigenvalue with modulus less than one and two with modulus greater than one. For concreteness, assume $|\lambda_1| < 1$, $|\lambda_2| > 1$, and $|\lambda_3| > 1$. If all eigenvalues are real $\mathbf{J}_u = \lambda_1$, $\mathbf{J}_s = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix}$ and $\mathbf{q}^{(k)}$ corresponds to the eigenvector associated with eigenvalue k, for $k = 1, 2, 3$. Otherwise, if $\lambda_2 = a + bi$ and $\lambda_3 = a - bi$ are complex conjugates, $\mathbf{J}_u = \lambda_1$, $\mathbf{J}_s = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\mathbf{q}^{(2)}$ and $\mathbf{q}^{(3)}$ are, respectively, the imaginary and real components of the eigenvector associated with the complex eigenvalues, while $\mathbf{q}^{(1)}$ is the eigenvector associated with λ_1.

Thus, $\mathbf{v}_t^u = 0$ all t. In addition $\mathbf{v}_t^s = \mathbf{J}_s E_t\{\mathbf{v}_{t+1}^s\}$, which has a stable solution:

$$\mathbf{v}_t^s = \mathbf{J}_s^{-1}\mathbf{v}_{t-1}^s + \boldsymbol{\xi}_t$$

where $\boldsymbol{\xi}_t$ is a (bivariate) martingale-difference process.

Thus, we have $\mathbf{x}_t = [\mathbf{q}^{(2)}, \mathbf{q}^{(3)}]\mathbf{v}_t^s$ is the sunspot solution. We can normalize the third element of $\mathbf{q}^{(2)}$ and $\mathbf{q}^{(3)}$ to unity.

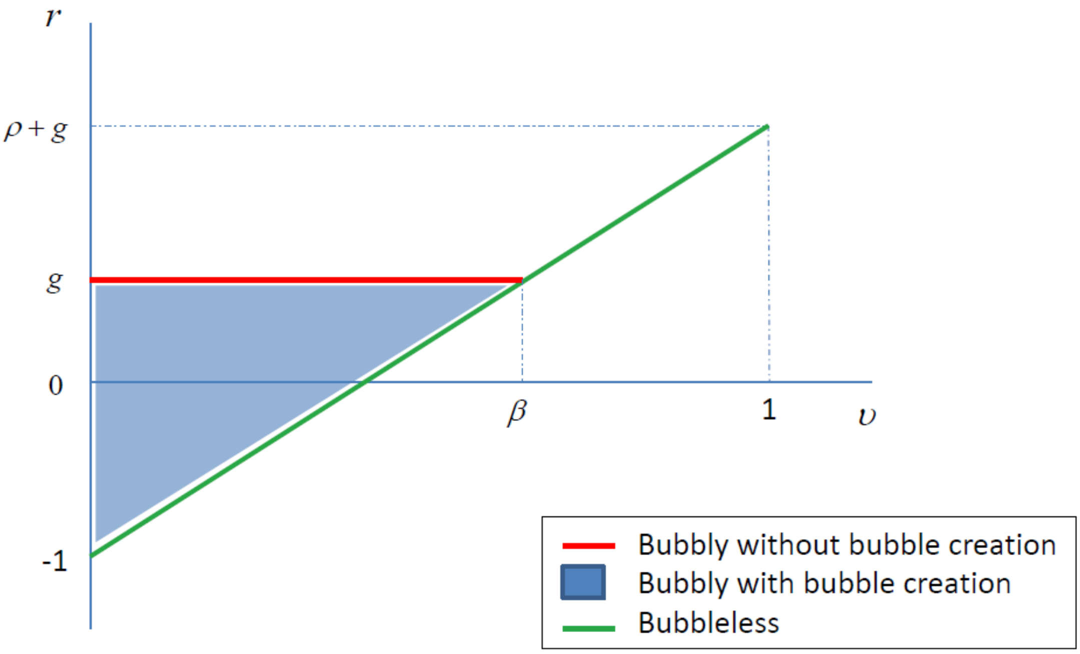

Figure 1. Balanced Growth Paths

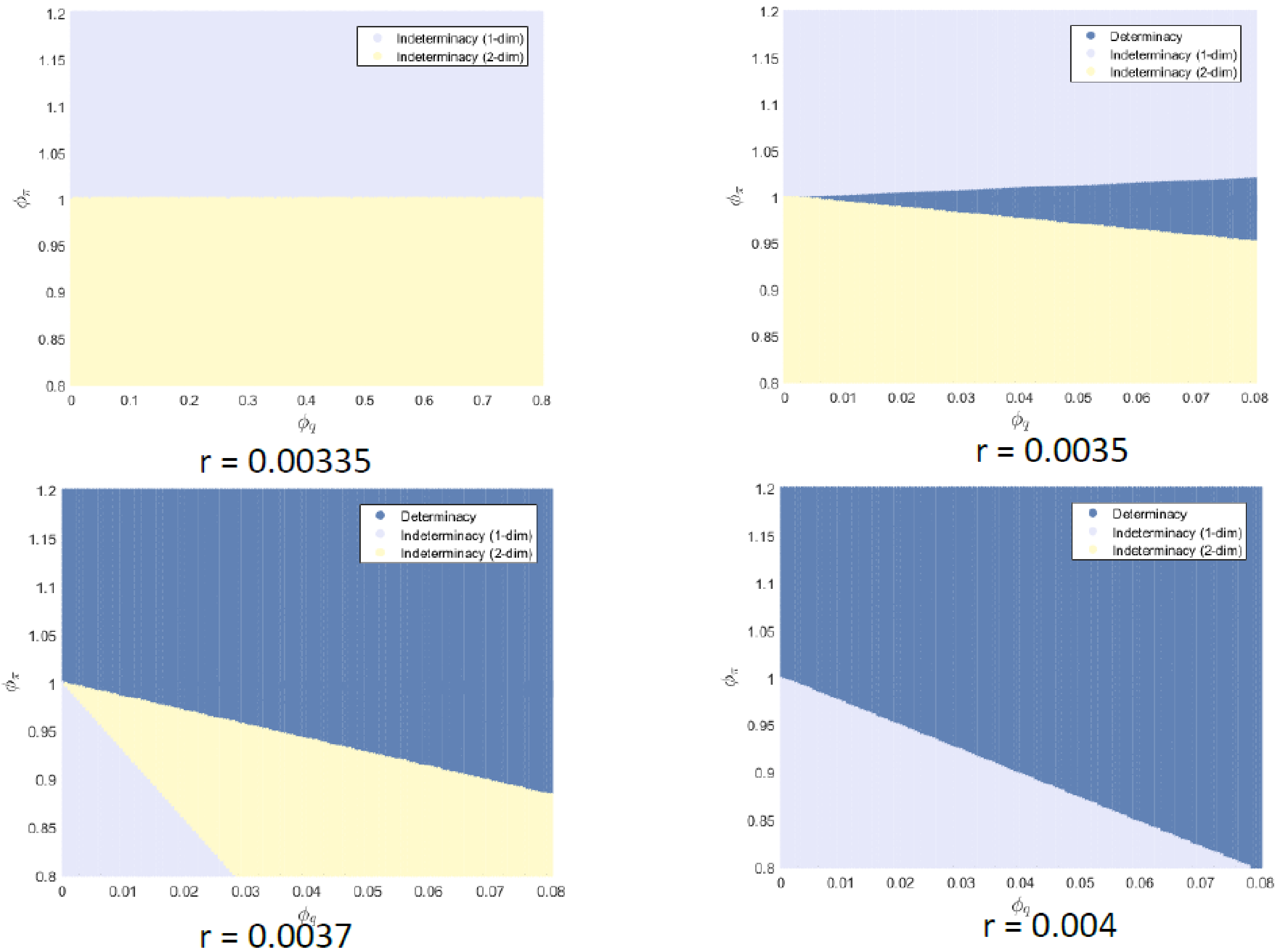

Figure 2. Determinacy and Indeterminacy Regions

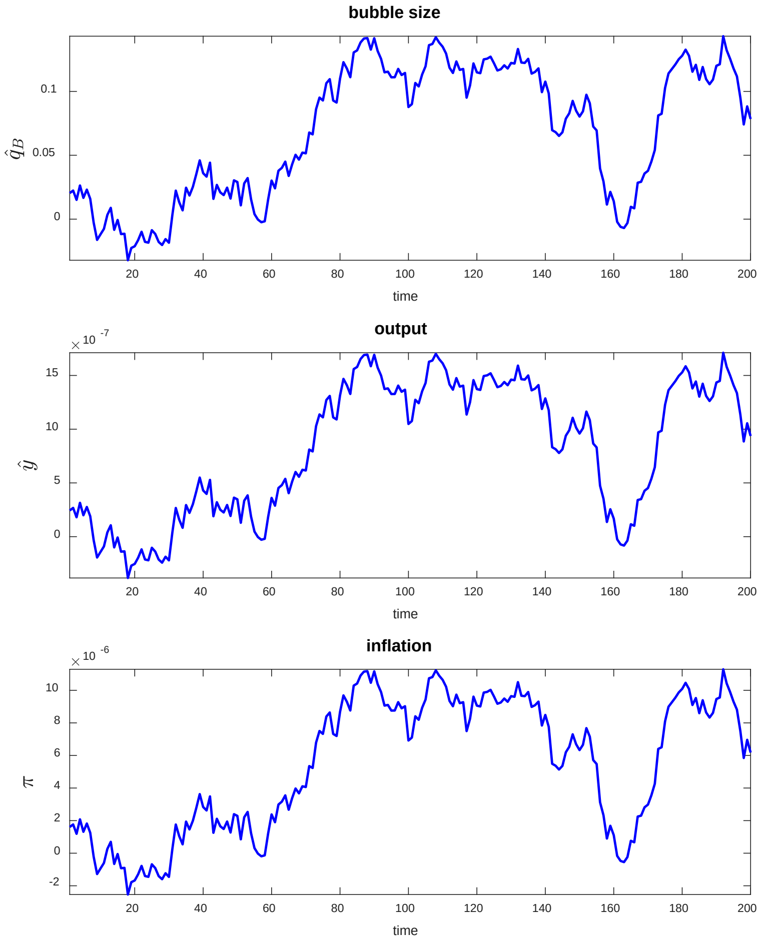

Figure 3. Simulated Bubble-driven Fluctuations around a Bubbly BGP

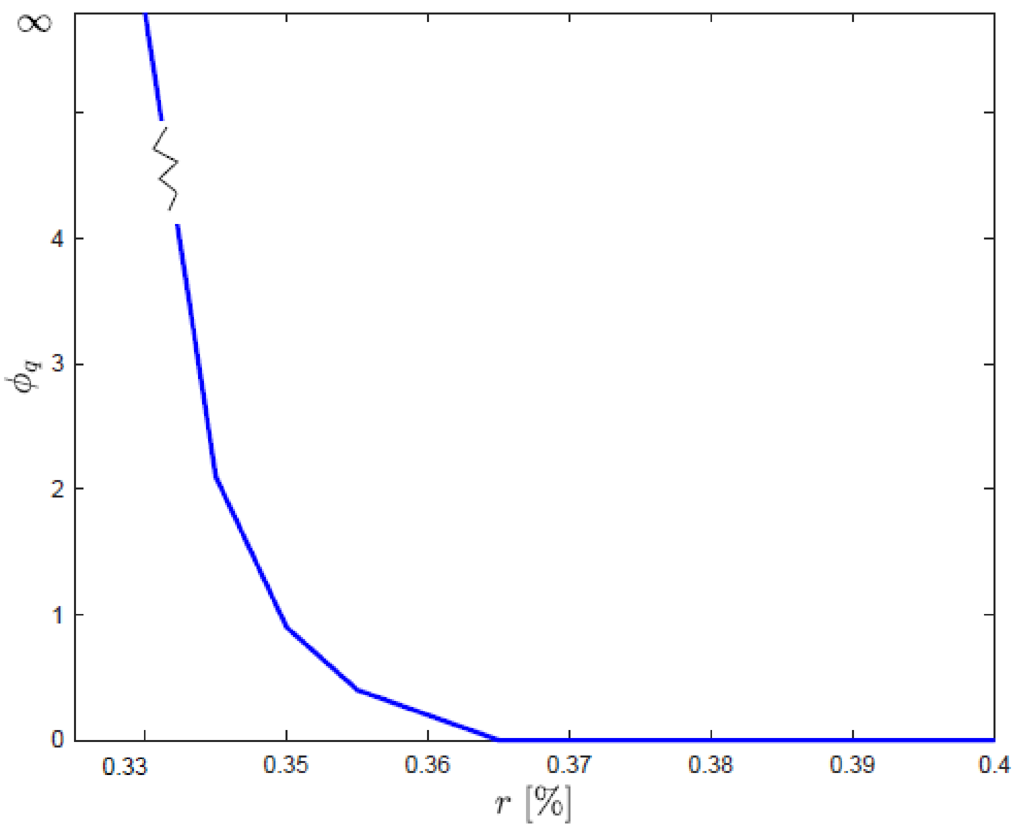

Figure 4. The Effectiveness of "Leaning against the Bubble" Policies

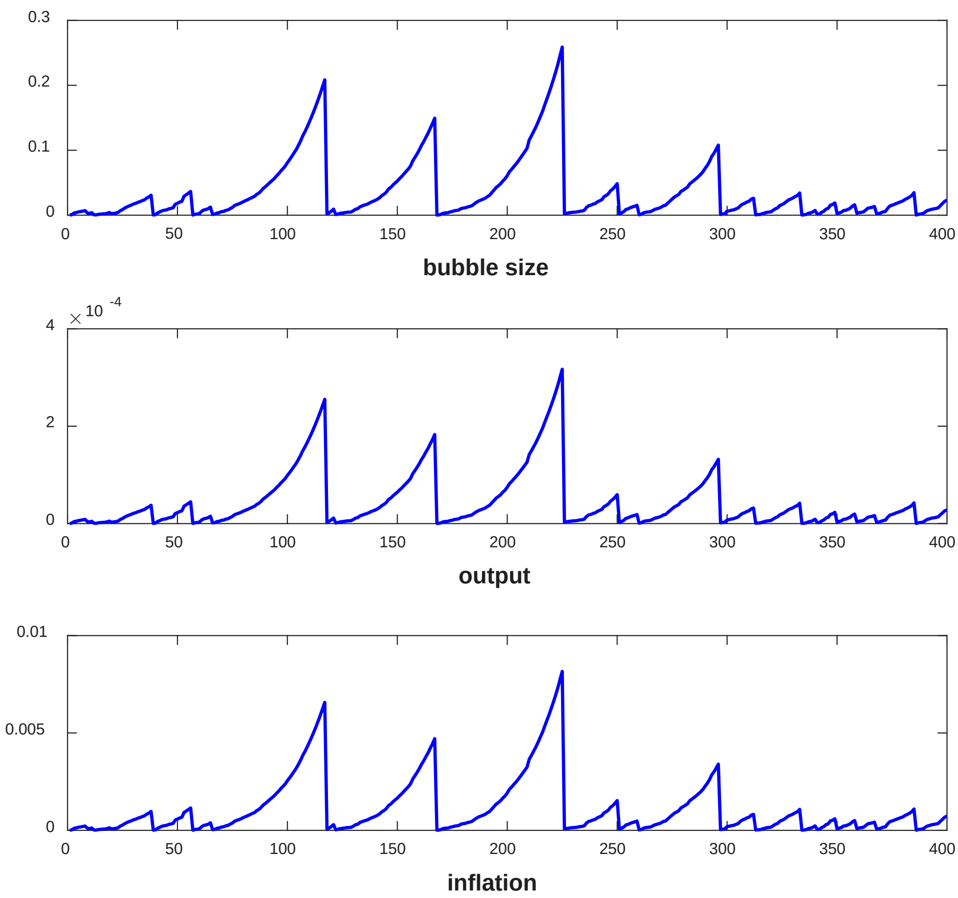

Figure 5. Simulated Bubble-Driven Fluctuations around the Bubbleless BGP

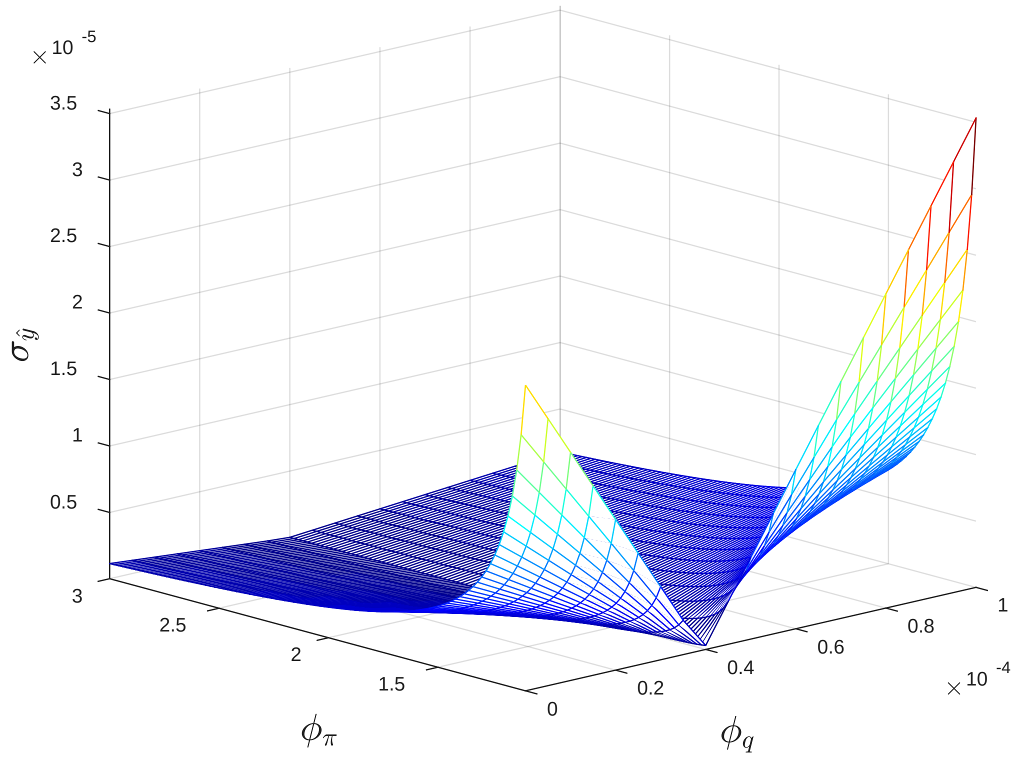

Figure 6. Bubble-Driven Fluctuations:

Monetary Policy and Macro Volatility

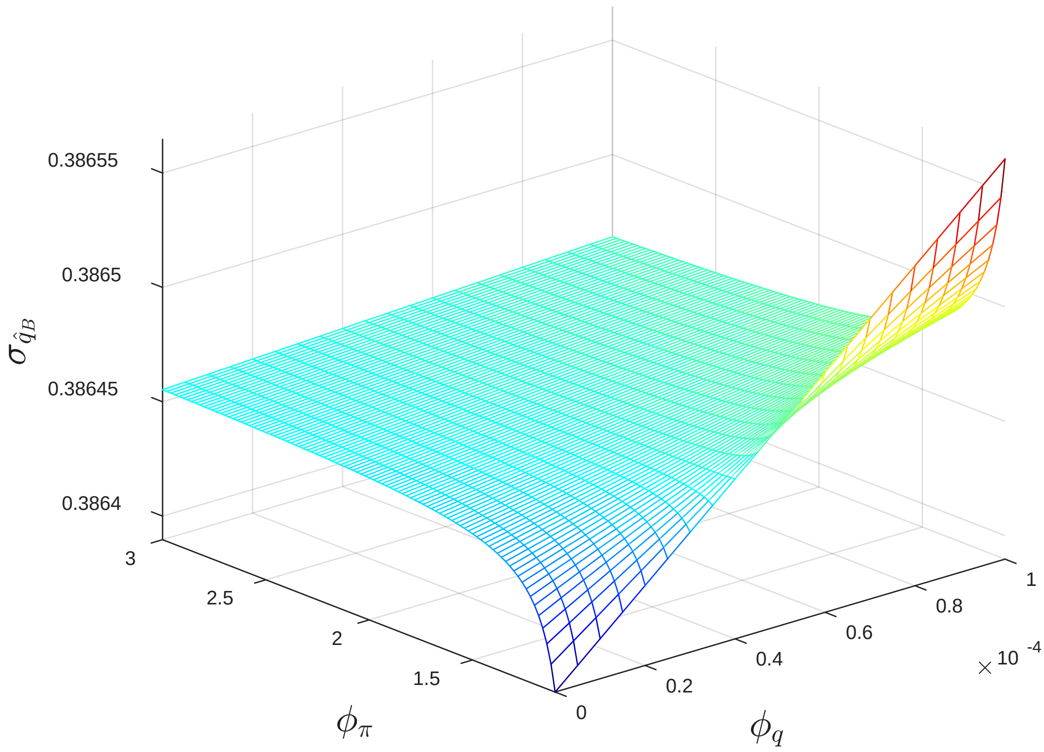

Figure 7. Bubble-Driven Fluctuations:

Monetary Policy and Bubble Volatility